ADI Finite Element Method for 2D Nonlinear Time Fractional Reaction-Subdiffusion Equation

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Abstract

In this paper, an alternating direction Galerkin finite element method is presented for solving 2D time fractional reaction sub-diffusion equation with nonlinear source term. Firstly, one order implicit-explicit method is used for time discretization, then Galerkin finite element method is adopted for spatial discretization and obtain a fully discrete linear system. Secondly, Galerkin alternating direction procedure for the system is derived by adding an extra term. Finally, the stability and convergence of the method are analyzed rigorously. Numerical results confirm the accuracy and efficiency of the proposed method.

Keywords

Nonlinear Fractional Differential Equation, Alternating Direction Implicit Method, Finite Element Method, Riemann-Liouville Fractional Derivative

1. Introduction

In this paper, we consider the following two-dimensional nonlinear fractional reaction-subdiffusion equation

$$\frac{\partial u(x,t)}{\partial t} = {}_0D_t^{1-\alpha}\left[\nabla\cdot\left(A\nabla u(x,t)\right) - \kappa u(x,t)\right] + f(u,x,t)$$
$$(x,t)\in\Omega_T \equiv \Omega\times(0,T],$$

(1)

with boundary and initial conditions

$$\begin{cases} u(x,t) = 0 & (x,t)\in\partial\Omega\times(0,T], \\ u(x,0) = \phi(x) & x\in\Omega, \end{cases}$$

(2)

where $0 < \alpha < 1$, $A = \text{diag}(\kappa_1, \kappa_2)$, $\phi(x)$ is sufficiently smooth function. For sim-

plicity, we assume coefficients κ_1, κ_2 and κ are positive constants in this paper. In fact, our method and its corresponding theoretic result are also valid for variable coefficients. $_0D_t^{1-\alpha}u(x,t)$ is the Riemann-Liouville time fractional derivative of order $1-\alpha$ defined by [1]

$$_0D_t^{1-\alpha}u(x,t) = \frac{\partial}{\partial t}\Big[_0D_t^{-\alpha}u(x,t)\Big],\tag{3}$$

where $_0D_t^{-\alpha}$ denotes the Riemann-Liouville fractional integral operator defined as [1]

$$_0D_t^{-\alpha}u(x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t u(x,s)(t-s)^{\alpha-1}\,\mathrm{d}s, \quad \alpha > 0.\tag{4}$$

In addition, we assume that the nonlinear source term $f(u,x,t)$ satisfies the Lipschitz condition with respect to u, *i.e.*, there exists a positive constant L such that

$$\big| f(u_1,x,t) - f(u_2,x,t)\big| \le L\big|u_1 - u_2\big|, \quad \forall u_1, u_2 \in \mathbb{R}.$$

Problem (1) can be considered as a model for reaction-diffusion phenomena with anomalous diffusion, which has been widely applied in various fields of science and engineering. Generally, solutions of (1) can't be obtained by analytical approach. So, there are various numerical methods developed for solving (1). Li and Ding [2] proposed higher order finite difference methods for solving 1D linear reaction and anomalous-diffusion equations. Zhuang, Liu and Anh, *et al.* [3] presented an implicit finite element method for solving 1D nonlinear fractional reaction-subdiffusion process. Dehghan, Abbaszadeh and Mohebbi [4] analyzed a meshless Galerkin method with radial basis functions of 2D linear fractional reaction-subdiffusion process. Yu, Jiang and Xu [5] derived an implicit compact finite difference scheme for solving 2D nonlinear fractional reaction-subdiffusion equation.

Alternating direction implicit (ADI) method was proposed by Peaceman, Rachford and Douglas [6] [7] [8] in 1950's for multidimensional differential equations of integer order, which could reduce original multidimensional problem into a sequences of one-dimensional problems. Since the first ADI based finite difference (FD) scheme presented for 2D space fractional diffusion equation by Meerschaert, Scheffler and Tadjeran [9], there are many literatures about various multidimensional fractional differential equations numerically solved by ADI technique. The following problem is always discussed:

$$_0^C D_t^\alpha u = \kappa_1 \frac{\partial^2 u}{\partial x^2} + \kappa_2 \frac{\partial^2 u}{\partial y^2} + f(x,y,t),\tag{5}$$

where $_0^C D_t^\alpha u$ is Caputo fractional derivative of order s defined as [1]

$$_0^C D_t^\alpha u(x,y,t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\varsigma)^{m-\alpha-1}\frac{\partial^\alpha u}{\partial t^\alpha}(x,y,\varsigma)\,\mathrm{d}\varsigma, \quad m-1 < \alpha < m, m \in \mathbb{N}.$$

In the case of $0 < \alpha < 1$, Zhang and Sun [10] presented an ADI FE scheme and analyzed its stability and convergence property by energy method. Cui [11] constructed a compact ADI FD scheme and discussed its stability by Fourier method. In the case of

$1 < \alpha < 2$, Zhang, Sun and Zhao [12] proposed a Crank-Nicolson compact ADI FD scheme, where stability and convergence property are also proved by energy method. Wang and Vong [13] presented another compact ADI FD scheme, where the original equation was first transformed into an equivalent form and then discretized by compact FD scheme combining with ADI technique. Li, Xu and Luo [14] presented an ADI finite element (FE) scheme for Equation (5), where stability and L_2 error estimate are analyzed. ADI orthogonal spline collocation (OSC) scheme was developed by Fairweather, Yang and Xu, *et al.* [15] for solving Equation (5); stability and error estimate in various norms H^j $(j = 0, 1, 2)$ are given. The equivalent form of Equation (5) as following

$$\frac{\partial u}{\partial t} = {}_0 D_t^{1-\alpha} \left(\kappa_1 \frac{\partial^2 u}{\partial x^2} + \kappa_2 \frac{\partial^2 u}{\partial y^2} \right) + f(x, y, t), \quad 0 < \alpha < 1, \tag{6}$$

is also often studied. For instances, Cui [16] presented a compact ADI FD scheme, where the local truncation error was analyzed and the stability was discussed by the Fourier method. Furthermore, the author analyzed the cause of low time accuracy when $\alpha \in (0, 1/2)$ and gave a remedy. Zhang and Sun [17] proposed a Crank-Nicolson compact ADI FD scheme for Equation (6), where stability and two error estimates are proved rigorously by energy method. Yao, Sun and Wu, *et al.* [18] first transformed Equation (6) into an equivalent form of Caputo fractional derivative and then derived a compact ADI FD scheme. Their numerical experiments show better numerical performance than the scheme in [16]. Compact ADI FD schemes were also derived for solving 2D/3D linear time fractional convection-diffusion equations [19] [20] [21] and 2D linear time fractional diffusion equations of distributed-order [22] [23].

There are also lots of ADI based numerical methods for multidimensional space fractional differential equations. Fast iterative ADI FD schemes [24] [25] are designed for 2D/3D linear space fractional diffusion equations, which are first order accuracy in both time and space and have the advantage of low computational work and low memory storage. High order accurate ADI FD schemes are proposed for 2D linear space fractional diffusion equations [26] [27] and two-sided space fractional convection-diffusion equations [28], which are based on weighted and shifted Grünwald operators or Lubich operators approximating Riemann-Liouville fractional derivatives respectively. Spectral direction splitting methods [29] are derived for 2D space fractional differential equations. Semi-implicit alternating direction FD scheme [30] and ADI FE scheme [31] are used for solving 2D fractional Fitz Hugh-Nagumo monodomain model, which consists of a coupled 2D space fractional nonlinear reaction-diffusion equation and an ordinary differential equation, on irregular domain and rectangle domain respectively. ADI Galerkin-Legendre spectral method [32] is developed for 2D Riesz space fractional nonlinear reaction-diffusion equation.

Most of the above mentioned works contribute on linear fractional differential equations and finite difference method combined with ADI technique. A few work consider ADI FEM [14] [31] or nonlinear fractional differential equations [30] [31] [32]. Compared with FD method, FE method has the advantage of easily handling variable coeffi-

cients problem and boundary conditions. And many realistic problems involve nonlinear fractional differential equations. Based on these motivations, our attention in this paper will focus on developing ADI FE schemes for efficiently solving a class of nonlinear time fractional differential equations. This is the first time ADI FE scheme proposed and analyzed rigorously for nonlinear fractional differential equations. We will use problem (1.1) as a model problem to illustrate our approach.

The outline of this paper is organized as follows. In Section 2, we introduce some preliminaries and notations which will be used later. The formulation of ADI finite element method for nonlinear time fractional reaction-subdiffusion equation is presented in Section 3. The stability and error estimates of the proposed method are discussed in Section 4. In convenience of computation, we give the matrix form of ADI finite element scheme in Section 5. Some numerical experiments are displayed in Section 6. It aims to confirm our theoretical results. In the end, some concluding remarks are given in Section 5.

In the following, C and C_i denote generic positive constants independent of τ, N and n, and their value will not be the same in different equations or inequalities.

2. Preliminary and Notations

Let Ω be a bounded and open domain in $\mathbb{R}^2$. Denote the inner product and norm on $L^2(\Omega)$ by

$$(f,g) = \int_\Omega f(x)g(x)\mathrm{d}x, \|f\|^2 = (f,f).$$

Recall that the Sobolev space $H^k(\Omega)$ is the closure of $C^\infty(\Omega)$ in the norm

$$\|v\|_k = \left(\sum_{0 \le \gamma_1 + \gamma_2 \le k} \left\| \frac{\partial^{\gamma_1 + \gamma_2} v}{\partial x^{\gamma_1} \partial y^{\gamma_2}} \right\|^2 \right)^{1/2}.$$

Denote by $H_0^1(\Omega)$ the closure of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_1$; it is well known that an equivalent norm on $H_0^1(\Omega)$ is

$$\|v\|_{H_0^1} = \|\nabla v\| = \left(\left\| \frac{\partial v}{\partial x} \right\|^2 + \left\| \frac{\partial v}{\partial y} \right\|^2 \right)^{1/2}.$$

If X is a normed space with norm $\|\cdot\|_X$, recall that

$$L^2(0,T;X) = \left\{ v : [0,T] \to X \, \|v\|_{L^2(X)} = \left(\int_0^T \|v(t)\|_X^2 \, \mathrm{d}t \right)^{1/2} < \infty \right\},$$

and

$$L^\infty(0,T;X) = \left\{ v : [0,T] \to X \, \|v\|_{L^\infty(X)} = \sup_{0 \le t \le T} \|v(t)\|_X < \infty \right\}.$$

Denote $\mathcal{Z} = \{v | v, v_x, v_y, v_{xy} \in L^2(\Omega)\}$. Let $\mathcal{M}_{h,r} (r \ge 2)$ be the finite dimensional subspace of $H_0^1(\Omega)$ and satisfy

$$\mathcal{M}_{h,r} \subset \mathcal{Z} \cap H_0^1(\Omega),$$

and

$$\inf_{\chi \in \mathcal{M}_{h,r}} \left(\sum_{m=0}^{2} h^m \sum_{\substack{i,j=0,1, \\ i+j=m}} \left\| \frac{\partial^m (v - \chi)}{\partial x^i \partial y^j} \right\| \right) \le Ch^s \|v\|_s, \quad v \in H^s(\Omega) \cap \mathcal{Z} \cap H_0^1(\Omega), \ 2 \le s \le r.$$

In convenience, the following notations will be used. For a positvie integer N, let $t_n = n\tau$, $n = 0, 1, \cdots, N$ where $\tau = T/N$ is the time step. Denote the grid function

$$u(x, t_n) = u^n, \ f(u^n, x, t_n) = f(u^n).$$

Define linear operator $\mathcal{L} : H_0^1(\Omega) \to L^2(\Omega)$ by

$$\mathcal{L}u := -\nabla \cdot (A\nabla u) + \kappa u, \tag{7}$$

and its variational form

$$(\mathcal{L}u, v)_A := (A\nabla u, \nabla v) + \kappa(u, v),$$

and corresponding energy norm by

$$\|u\|_A := (\mathcal{L}u, u)_A^{1/2} = \left(\left\| A^{1/2} \nabla u \right\|^2 + \kappa \|u\|^2 \right)^{1/2}.$$

Obviously, we have

$$(\mathcal{L}u, v)_A \le \|u\|_A^2 \|v\|_A^2.$$

Some useful lemmas are given as follows.

Lemma 1. [3] *Let*

$$w_j = (j+1)^\alpha - j^\alpha, \ j = 0, 1, 2,$$

then w_j *satisfy the following properties*
1) $w_0 = 1, w_j > 0, \ j = 0, 1, 2, \cdots,$
2) $w_j > w_{j+1}, \ j = 0, 1, 2, \cdots.$

Lemma 2. [3] *If* $y(t) \in C^2[0, T]$, *then*

$$_0 D_t^{-\alpha} y(t_{n+1}) - {}_0 D_t^{-\alpha} y(t_n) = \mu \left[y(t_{n+1}) + \sum_{j=0}^{n-1} \left(w_{j+1} - w_j \right) y(t_{n-j}) \right] + R_\alpha^n,$$

$$0 \le n \le N-1$$

where $\mu = \tau^\alpha / \Gamma(\alpha+1)$, R_α^n *is bounded by*

$$\left| R_\alpha^n \right| \le C w_n \tau^{1+\alpha}.$$

We state here for convenience the discrete version of Gronwall's inequality.

Lemma 3. [33] *Suppose that* ϕ, ψ *and* χ *are nonnegative functions defined for* $t = n\tau, n = 0, 1, \cdots, N$, *and that* χ *is nondecreasing. If*

$$\phi_k + \psi_k \le \chi_k + C\tau \sum_{n=0}^{k-1} \phi_n, \quad k = 0, 1, \cdots, N,$$

where C *is a positive constant, then*

$$\phi_k + \psi_k \le \chi_k e^{Ck\tau}, \quad k = 0, 1, \cdots, N.$$

3. Formulation of ADI FEM

Integrating both sides of (1) with respect to the time variable t from t_n to t_{n+1}, and noticing the definition (3) and (7), we can obtain

$$u^{n+1} - u^n = {}_0D_t^{-\alpha}\left[-\mathcal{L}u\right](t_{n+1}) - {}_0D_t^{-\alpha}\left(-\mathcal{L}u\right)(t_n) + \int_{t_n}^{t_{n+1}} f(u,x,t)\,\mathrm{d}t. \tag{8}$$

Applying Lemma 2 and the following integration formula

$$\int_{t_n}^{t_{n+1}} f(u,x,t)\,\mathrm{d}t = \tau f(u^n) + \mathcal{O}(\tau^2),$$

Equation (8) is equivalent to

$$u^{n+1} + \mu\left[\mathcal{L}u^{n+1} + \sum_{j=0}^{n-1}\left(w_{j+1} - w_j\right)\mathcal{L}u^{n-j}\right] = u^n + \tau f(u^n) + R_\alpha, \tag{9}$$

where the remainder term $R_\alpha = R_\alpha^n + \mathcal{O}(\tau^2)$ can be bounded by

$$\left|R_\alpha\right| \le C w_n \tau^{1+\alpha}. \tag{10}$$

The weak form of Equation (9) is: find $u^{n+1} \in H_0^1(\Omega)$ such that for any $v \in H_0^1(\Omega)$,

$$\left(u^{n+1}, v\right) + \mu\left[\left(\mathcal{L}u^{n+1}, v\right)_A + \sum_{j=0}^{n-1}\left(w_{j+1} - w_j\right)\left(\mathcal{L}u^{n-j}, v\right)_A\right]$$
$$= \left(u^n, v\right) + \tau\left(f(u^n), v\right) + \left(R_\alpha, v\right), \quad 0 \le n \le N-1. \tag{11}$$

Then the finite element approximation to Equation (11) is: find $U^n \in \mathcal{M}_{h,r}$ such that for any $V \in \mathcal{M}_{h,r}$,

$$\left(U^{n+1}, V\right) + \mu\left[\left(\mathcal{L}U^{n+1}, V\right)_A + \sum_{j=0}^{n-1}\left(w_{j+1} - w_j\right)\left(\mathcal{L}U^{n-j}, V\right)_A\right]$$
$$= \left(U^n, V\right) + \tau\left(f(U^n), V\right), \quad 0 \le n \le N-1,$$

or equivalently

$$\left(1 + \mu\kappa\right)\left(U^{n+1}, V\right) + \mu\left(A\nabla U^{n+1}, \nabla V\right)$$
$$= \left(U^n, V\right) + \mu\sum_{j=0}^{n-1}\left(w_j - w_{j+1}\right)\left[\left(A\nabla U^{n-j}, \nabla V\right) + \kappa\left(U^{n-j}, V\right)\right] + \left(\tau f(U^n), V\right), \tag{12}$$
$$0 \le n \le N-1.$$

The choice of the initial values U^0 will be discussed later.

Let $E^{n+1} = U^{n+1} - U^n$, then $U^{n+1} = E^{n+1} + U^n$, (12) can be transformed into

$$\left(1 + \mu\kappa\right)\left(E^{n+1}, V\right) + \mu\left(A\nabla E^{n+1}, \nabla V\right)$$
$$= -\mu\kappa\left(U^n, V\right) - \mu\left(A\nabla U^n, \nabla V\right)$$
$$+ \mu\sum_{j=0}^{n-1}\left(w_j - w_{j+1}\right)\left[\left(A\nabla U^{n-j}, \nabla V\right) + \kappa\left(U^{n-j}, V\right)\right] + \left(\tau f(U^n), V\right), \tag{13}$$
$$0 \le n \le N-1,$$

so that the alternating-direction Galerkin scheme of (1) can be defined as, for $n = 0, \cdots, N-1$,

$$\left(E^{n+1},V\right)+\frac{\mu}{1+\mu\kappa}\left(A\nabla E^{n+1},\nabla V\right)+\frac{\mu^2\kappa_1\kappa_2}{\left(1+\mu\kappa\right)^2}\left(\frac{\partial^2 E^{n+1}}{\partial x\partial y},\frac{\partial^2 V}{\partial x\partial y}\right)=\Psi^n\left(V\right),\forall V\in\mathcal{M}_{h,r},\quad(14)$$

where

$$\Psi^n\left(V\right)=\frac{1}{1+\mu\kappa}\left[-\mu\kappa\left(U^n,V\right)-\mu\left(A\nabla U^n,\nabla V\right)\right]$$
$$+\frac{\mu}{1+\mu\kappa}\sum_{j=0}^{n-1}\left(w_j-w_{j+1}\right)\left[\left(A\nabla U^{n-j},\nabla V\right)+\kappa\left(U^{n-j},V\right)\right]+\frac{\tau}{1+\mu\kappa}\left(f\left(U^n\right),V\right).\quad(15)$$

Remark 1. Numerical experiments in Section 6 demonstrate that the ADI Galerkin finite element scheme (14) has bad numerical performance for $0<\alpha<1/2$. Similar phenomena have been reported in [11] [16] and [17], where compact ADI finite difference schemes are designed respectively for solving 2D linear time fractional sub-diffusion equations. To construct ADI finite element method, the third term

$$\frac{\mu^2\kappa_1\kappa_2}{\left(1+\mu\kappa\right)^2}\left(\frac{\partial^2 E^{n+1}}{\partial x\partial y},\frac{\partial^2 V}{\partial x\partial y}\right)$$

in left hand side of (14) is extra added. Its effect on temporal accuracy cannot be ignored when $0<\alpha<1/2$. To balance it, we can add a correction term

$$\frac{\mu^2\kappa_1\kappa_2}{\left(1+\mu\kappa\right)^2}\left(\frac{\partial^2 E^n}{\partial x\partial y},\frac{\partial^2 V}{\partial x\partial y}\right)\quad(16)$$

on the right hand side of (14). By the way, the similar remedy was also adopted by [11] [16] in finite difference framework.

In conclusion, when $0<\alpha<1/2$, it is beneficial to use the following scheme for $n\geq 1$

$$\left(E^{n+1},V\right)+\frac{\mu}{1+\mu\kappa}\left(A\nabla E^{n+1},\nabla V\right)+\frac{\mu^2\kappa_1\kappa_2}{\left(1+\mu\kappa\right)^2}\left(\frac{\partial^2 E^{n+1}}{\partial x\partial y},\frac{\partial^2 V}{\partial x\partial y}\right)=\tilde{\Psi}^n\left(V\right),\quad\forall V\in\mathcal{M}_{h,r},\quad(17)$$

where

$$\tilde{\Psi}^n\left(V\right)=\Psi^n\left(V\right)+\frac{\mu^2\kappa_1\kappa_2}{\left(1+\mu\kappa\right)^2}\left(\frac{\partial^2 E^n}{\partial x\partial y},\frac{\partial^2 V}{\partial x\partial y}\right),$$

Ψ^n is defined by (15). And for $n=0$ the approximation is still given by (14).

In the following, we will focus our attention on the ADI Galerkin finite element scheme (14).

4. Stability and Error Estimate

Firstly, we introduce some notation and lemmas. Given a smooth function $u(x,t)$, define $W:[0,T]\to\mathcal{M}_{h,r}$ by

$$\left(A\nabla\left(u-W\right),\nabla V\right)+\kappa\left(u-W,V\right)=0,\quad\forall V\in\mathcal{M}_{h,r},\quad(18)$$

or equivalently

$$\left(\mathcal{L}\left(u-W\right),V\right)_A=0,\quad\forall V\in\mathcal{M}_{h,r}.$$

The operator defined in (18) has the following approximate properties:

Lemma 4. [34] *Let* $\eta = u - W$ *for* $k = 0, 1, 2$, $p = 2, \infty$, $j = 0, 1$ *there exists a constant* C, *independent of meshsize* h, *such that*

$$\left\| \frac{\partial^k \eta}{\partial t^k} \right\|_{L^p(H^j)} \le Ch^{s-j} \left\| \frac{\partial^k u}{\partial t^k} \right\|_{L^p(H^s)}, \quad 1 \le s \le r. \tag{19}$$

Lemma 5. [33] *If* $\mathcal{D}$ *denote the operators* $\mathrm{d}_t, \dfrac{\partial}{\partial t}$ *and* $\dfrac{\partial^2}{\partial t^2}$, *then*

$$\left\| \frac{\partial^2 (\mathcal{D}\eta^n)}{\partial x \partial y} \right\| \le Ch^{r-2} \|\mathcal{D}u\|_r + Ch^{-2} \|\mathcal{D}\eta^n\|, \tag{20}$$

where $\mathrm{d}_t v^n = (v^n - v^{n-1})/\tau$.

Lemma 6. *Let* $\mathcal{L}$ *and* w_j *are defined in* (7) *and Lemma* 1, *respectively. For any positive integer* n *and* $v \in H_0^1(\Omega)$, *let*

$$\mathcal{T}(v^{n+1}) := (\mathcal{L}v^{n+1}, v^{n+1})_A + \sum_{j=0}^{n-1} (w_{j+1} - w_j)(\mathcal{L}v^{n-j}, v^{n+1})_A, \quad n = 0, 1, \cdots \tag{21}$$

then it holds that

$$\sum_{n=0}^{k-1} \mathcal{T}(v^{n+1}) = \frac{1}{2} \sum_{n=0}^{k-1} (w_n + w_{k-n-1}) \|v^{n+1}\|_A^2 \ge 0, \quad \forall k \ge 1.$$

Proof. When $n = 0$, it is easily get $\mathcal{T}(v^1) = \|v^1\|_A^2 \ge 0$. Now we consider $n \ge 1$. Note that

$$(\mathcal{L}v^{n+1}, v^{n+1})_A = \|v^{n+1}\|_A^2,$$

$$(\mathcal{L}v^{n-j}, v^{n+1})_A \le \frac{1}{2} \left(\|v^{n-j}\|_A^2 + \|v^{n+1}\|_A^2 \right)$$

and $w_j > w_{j+1}$, we get

$$\mathcal{T}(v^{n+1}) \ge \|v^{n+1}\|_A^2 - \frac{1}{2} \sum_{j=0}^{n-1} (w_j - w_{j+1}) \left(\|v^{n-j}\|_A^2 + \|v^{n+1}\|_A^2 \right)$$

$$= \frac{1}{2}(1 + w_n) \|v^{n+1}\|_A^2 - \frac{1}{2} \sum_{j=0}^{n-1} (w_j - w_{j+1}) \|v^{n-j}\|_A^2.$$

By direct algebraic calculation, and noticing the properties of w_j in Lemma 1, we obtain

$$\sum_{n=0}^{k-1} \mathcal{T}(v^{n+1}) \ge \|v^1\|_A^2 + \frac{1}{2} \sum_{n=1}^{k-1} (1 + w_n) \|v^{n+1}\|_A^2 - \frac{1}{2} \sum_{n=1}^{k-1} \sum_{j=0}^{n-1} (w_j - w_{j+1}) \|v^{n-j}\|_A^2$$

$$= \frac{1}{2} \sum_{n=0}^{k-1} (1 + w_n) \|v^{n+1}\|_A^2 - \frac{1}{2} \sum_{n=0}^{k-2} (1 - w_{k-n-1}) \|v^{n+1}\|_A^2$$

$$= \frac{1}{2} \sum_{n=0}^{k-1} (w_n + w_{k-n-1}) \|v^{n+1}\|_A^2 \ge 0.$$

The proof is completed.

Next, we consider the stability of the ADI Galerkin method (14). Define the following problem dependent norm for any $v \in \mathcal{Z} \cap H_0^1(\Omega)$,

$$\interleave v \interleave := \left(\| v \|^2 + \frac{\mu^2 \kappa_1 \kappa_2}{1 + \mu\kappa} \left\| \frac{\partial^2 v}{\partial x \partial y} \right\|^2 \right)^{1/2}, \tag{22}$$

which will be used in stability analysis.

Assume the initial value U^0 has an error $\delta U^0 \in \mathcal{M}_{h,r}$, which will cause the solution of ADI Galerkin scheme (14), $U^{k+1} (0 \le k \le N-1)$, has a perturbation $\delta U^{k+1} \in \mathcal{M}_{h,r}$. Then the stability property of ADI Galerkin methods can be represented as following:

Theorem 1. *The ADI Galerkin method* (14) *is stable with respect to initial value* U^0 *in the problem dependent norm* (22), *i.e., there exist a positive constant* Θ *which independent of* k *such that*

$$\interleave \delta U^{k+1} \interleave \le \Theta \interleave \delta U^0 \interleave$$

holds for any $0 \le k \le N-1$.

Proof. The equivalent form of (14) is:

$$\left(U^{n+1}, V \right) + \mu \left[\left(\mathcal{L} U^{n+1}, V \right)_A + \sum_{j=0}^{n-1} \left(w_{j+1} - w_j \right) \left(\mathcal{L} U^{n-j}, V \right)_A \right] + \frac{\tau \mu^2 \kappa_1 \kappa_2}{1 + \mu\kappa} \left(\frac{\partial^2 \left[d_t U^{n+1} \right]}{\partial x \partial y}, \frac{\partial^2 V}{\partial x \partial y} \right) \tag{23}$$

$$= \left(U^n, V \right) + \tau \left(f \left(U^n \right), V \right).$$

Then the perturbation equation of (23) can be written as

$$\left(\delta U^{n+1}, V \right) + \mu \left[\left(\mathcal{L} \delta U^{n+1}, V \right)_A + \sum_{j=0}^{n-1} \left(w_{j+1} - w_j \right) \left(\mathcal{L} \delta U^{n-j}, V \right)_A \right]$$

$$+ \frac{\tau \mu^2 \kappa_1 \kappa_2}{1 + \mu\kappa} \left(\frac{\partial^2 \left[d_t \delta U^{n+1} \right]}{\partial x \partial y}, \frac{\partial^2 V}{\partial x \partial y} \right) = \left(\delta U^n, V \right) + \tau \left(\delta f^n, V \right), \tag{24}$$

where $\delta f^n = f \left(U^n + \delta U^n \right) - f \left(U^n \right)$ and $V \in \mathcal{M}_{h,r}$.

Taking $V = 2 \delta U^{n+1}$ in (24), employing Schwartz inequality and the notation $\mathcal{T} (\cdot)$, yields

$$\left\| \delta U^{n+1} \right\|^2 - \left\| \delta U^n \right\|^2 + 2\mu \mathcal{T} \left(\delta U^{n+1} \right) + \frac{\mu^2 \kappa_1 \kappa_2}{1 + \mu\kappa} \left(\left\| \frac{\partial^2}{\partial x \partial y} \left(\delta U^{n+1} \right) \right\|^2 - \left\| \frac{\partial^2}{\partial x \partial y} \left(\delta U^n \right) \right\|^2 \right)$$

$$\le 2\tau \left\| \delta f^n \right\| \cdot \left\| \delta U^{n+1} \right\|. \tag{25}$$

Further, by Lipschitz property of f and Schwartz inequality, we can estimate the right hand side of (25) as

$$2\tau \left\| \delta f^n \right\| \cdot \left\| \delta U^n \right\| \le 2\tau L \left\| \delta U^n \right\| \left\| \delta U^{n+1} \right\| \le \tau L \left\| \delta U^n \right\|^2 + \tau L \left\| \delta U^{n+1} \right\|^2. \tag{26}$$

Summing for $n = 0, 1, \cdots, k$ in (25) and using (26) and Lemma 6, we can write

$$\left\| \delta U^{k+1} \right\|^2 + \frac{\mu^2 \kappa_1 \kappa_2}{1 + \mu\kappa} \left\| \frac{\partial^2}{\partial x \partial y} \left(\delta U^{k+1} \right) \right\|^2$$

$$\le \left\| \delta U^0 \right\|^2 + \frac{\mu^2 \kappa_1 \kappa_2}{1 + \mu\kappa} \left\| \frac{\partial^2}{\partial x \partial y} \left(\delta U^0 \right) \right\|^2 + \tau L \sum_{n=0}^{k} \left\| \delta U^n \right\|^2 + \tau L \sum_{n=0}^{k} \left\| \delta U^{n+1} \right\|^2. \tag{27}$$

For sufficiently small τ, it follows from (27) that

$$\left\|\delta U^{k+1}\right\|^2 + \frac{\mu^2 \kappa_1 \kappa_2}{1+\mu\kappa}\left\|\frac{\partial^2}{\partial x \partial y}\left(\delta U^{k+1}\right)\right\|^2 \le C_1 \left\|\delta U^0\right\|^2 + C_2 \tau \sum_{n=1}^{k}\left\|\delta U^n\right\|^2.$$

Using discrete Gronwall inequality, we have

$$\left\|\delta U^{k+1}\right\|^2 + \frac{\mu^2 \kappa_1 \kappa_2}{1+\mu\kappa}\left\|\frac{\partial^2}{\partial x \partial y}\left(\delta U^{k+1}\right)\right\|^2 \le C_1 \left\|\delta U^0\right\|^2 \cdot e^{C_2(k+1)\tau} \le C_1 \left\|\delta U^0\right\|^2 \cdot e^{C_2 T}.$$

Let $\Theta = \sqrt{C_1 e^{C_2 T}}$, we conclude $\left\|\delta U^{k+1}\right\| \le \Theta \left\|\delta U^0\right\|$. The proof is completed.

Theorem 2. *Let u and $\left\{U^n\right\}_{n=1}^{N}$ denote the solutions of* (1) *and* (14) *respectively. Assume that $u \in L^{\infty}\left(H^r\right), u_t \in L^2\left((0,T];H^r\right), u_{xyt} \in L^2\left((0,T];L^2\right)$, where $r \ge 2$. Then, for τ sufficiently small,*

$$\max_{0 \le n \le N}\left\|U^n - u^n\right\| \le C\left(\tau^{\alpha} + h^r\right),$$

provided the initial value U^0 satisfy

$$\left\|U^0 - W^0\right\| + \tau^{\alpha}\left\|\frac{\partial^2}{\partial x \partial y}\left(U^0 - W^0\right)\right\| \le Ch^r.$$

Proof. Denote $\xi^n = U^n - W^n, \eta^n = u^n - W^n$, then $U^n - u^n = \xi^n - \eta^n$. With W as in (18) we have from (11) that

$$\left(W^{n+1}, V\right) + \mu\left[\left(\mathcal{L}W^{n+1}, V\right)_A + \sum_{j=0}^{n-1}\left(w_{j+1} - w_j\right)\left(\mathcal{L}W^{n-j}, V\right)_A\right]$$
$$+ \frac{\tau\mu^2\kappa_1\kappa_2}{1+\mu\kappa}\left(\frac{\partial^2\left[d_t W^{n+1}\right]}{\partial x \partial y}, \frac{\partial^2 V}{\partial x \partial y}\right)$$
$$= \left(W^n, V\right) + \left(\tau f\left(u^n\right) + R_{\alpha}, V\right) - \tau\left(d_t \eta^{n+1}, V\right) \tag{28}$$
$$+ \frac{\tau\mu^2\kappa_1\kappa_2}{1+\mu\kappa}\left(\frac{\partial^2}{\partial x \partial y}\left(d_t u^{n+1} - d_t \eta^{n+1}\right), \frac{\partial^2 V}{\partial x \partial y}\right)$$

Subtracting (28) from (23) leads to

$$\left(\xi^{n+1}, V\right) + \mu\left[\left(\mathcal{L}\xi^{n+1}, V\right)_A + \sum_{j=0}^{n-1}\left(w_{j+1} - w_j\right)\left(\mathcal{L}\xi^{n-j}, V\right)_A\right]$$
$$+ \frac{\tau\mu^2\kappa_1\kappa_2}{1+\mu\kappa}\left(\frac{\partial^2\left[d_t \xi^{n+1}\right]}{\partial x \partial y}, \frac{\partial^2 V}{\partial x \partial y}\right)$$
$$= \left(\xi^n, V\right) + \tau\left(f\left(U^n\right) - f\left(u^n\right), V\right) - \left(R_{\alpha}, V\right) + \tau\left(d_t \eta^{n+1}, V\right) \tag{29}$$
$$+ \frac{\tau\mu^2\kappa_1\kappa_2}{1+\mu\kappa}\left(\frac{\partial^2}{\partial x \partial y}\left(d_t \eta^{n+1} - d_t u^{n+1}\right), \frac{\partial^2}{\partial x \partial y}V\right)$$

Taking $V = 2\xi^{n+1}$, by Schwartz inequality and using the notation $\mathcal{T}(\cdot)$, then (29) reduces to

$$\left\|\xi^{n+1}\right\|^2 - \left\|\xi^n\right\|^2 + 2\mu\mathcal{T}\left(\xi^{n+1}\right) + \frac{\mu^2\kappa_1\kappa_2}{1+\mu\kappa}\left(\left\|\frac{\partial^2\xi^{n+1}}{\partial x\partial y}\right\|^2 - \left\|\frac{\partial^2\xi^n}{\partial x\partial y}\right\|^2\right)$$

$$\leq 2\left(\tau L\left\|\xi^n - \eta^n\right\| + \left\|R_\alpha\right\| + \tau\left\|\mathrm{d}_t\eta^{n+1}\right\|\right)\cdot\left\|\xi^{n+1}\right\| \tag{30}$$

$$+ \frac{2\tau\mu^2\kappa_1\kappa_2}{1+\mu\kappa}\left\|\frac{\partial^2}{\partial x\partial y}\left(\mathrm{d}_t\eta^{n+1} - \mathrm{d}_t u^{n+1}\right)\right\|\cdot\left\|\frac{\partial^2\xi^{n+1}}{\partial x\partial y}\right\|$$

Now, by Young inequality and Schwartz inequality, we can write

$$2\left(\tau L\left\|\xi^n - \eta^n\right\| + \tau\left\|\mathrm{d}_t\eta^{n+1}\right\|\right)\cdot\left\|\xi^{n+1}\right\| \leq C\tau\left(\left\|\xi^n\right\|^2 + \left\|\eta^n\right\|^2 + \left\|\mathrm{d}_t\eta^{n+1}\right\|^2\right) + \tau\left\|\xi^{n+1}\right\|^2,$$

$$2\left\|R_\alpha\right\|\cdot\left\|\xi^{n+1}\right\| \leq \frac{\kappa}{\mu w_n}\left\|R_\alpha\right\|^2 + \frac{\mu w_n}{\kappa}\left\|\xi^{n+1}\right\|^2 \leq \frac{\kappa}{\mu w_n}\left\|R_\alpha\right\|^2 + \mu w_n\left\|\xi^{n+1}\right\|_A^2,$$

$$2\left\|\frac{\partial^2}{\partial x\partial y}\left(\mathrm{d}_t\eta^{n+1} - \mathrm{d}_t u^{n+1}\right)\right\|\cdot\left\|\frac{\partial^2\xi^{n+1}}{\partial x\partial y}\right\| \leq \left\|\frac{\partial^2}{\partial x\partial y}\left(\mathrm{d}_t\eta^{n+1} - \mathrm{d}_t u^{n+1}\right)\right\|^2 + \left\|\frac{\partial^2\xi^{n+1}}{\partial x\partial y}\right\|^2.$$

Substitute the above three inequalities into the right hand side of (30), and sum for $n = 0, 1, \cdots, N-1$. Employing Lemma 6, we can write

$$\left\|\xi^N\right\|^2 + \frac{\mu^2\kappa_1\kappa_2}{1+\mu\kappa}\left\|\frac{\partial^2\xi^N}{\partial x\partial y}\right\|^2$$

$$\leq C\left(\left\|\xi^0\right\|^2 + \mu^2\left\|\frac{\partial^2\xi^0}{\partial x\partial y}\right\|^2\right) + C\sum_{n=0}^{N-1}\tau\left(\left\|\eta^n\right\|^2 + \left\|\mathrm{d}_t\eta^{n+1}\right\|^2\right) + \sum_{n=0}^{N-1}\frac{\kappa}{\mu w_n}\left\|R_\alpha\right\|^2$$

$$+ C\sum_{n=0}^{N-1}\tau\mu^2\left\|\frac{\partial^2}{\partial x\partial y}\left(\mathrm{d}_t\eta^{n+1} - \mathrm{d}_t u^{n+1}\right)\right\|^2 + C\tau\sum_{n=1}^{N}\left\|\xi^n\right\|^2 + \tau\sum_{n=1}^{N}\frac{\mu^2\kappa_1\kappa_2}{1+\mu\kappa}\left\|\frac{\partial^2\xi^n}{\partial x\partial y}\right\|^2.$$

By discrete Gronwall inequality, if τ sufficiently small, we can estimate

$$\left\|\xi^N\right\|^2 + \frac{\mu^2\kappa_1\kappa_2}{1+\mu\kappa}\left\|\frac{\partial^2\xi^N}{\partial x\partial y}\right\|^2$$

$$\leq C\left(\left\|\xi^0\right\|^2 + \mu^2\left\|\frac{\partial^2\xi^0}{\partial x\partial y}\right\|^2\right) + C\sum_{n=0}^{N-1}\tau\left(\left\|\eta^n\right\|^2 + \left\|\mathrm{d}_t\eta^{n+1}\right\|^2\right)$$

$$+ \sum_{n=0}^{N-1}\frac{\kappa}{\mu w_n}\left\|R_\alpha\right\|^2 + C\sum_{n=0}^{N-1}\tau\mu^2\left\|\frac{\partial^2}{\partial x\partial y}\left(\mathrm{d}_t\eta^{n+1} - \mathrm{d}_t u^{n+1}\right)\right\|^2.$$

Consequently, we obtain

$$\left\|\xi^N\right\|^2 \leq C\left(\left\|\xi^0\right\|^2 + \mu^2\left\|\frac{\partial^2\xi^0}{\partial x\partial y}\right\|^2\right) + C\sum_{n=0}^{N-1}\tau\left(\left\|\eta^n\right\|^2 + \left\|\mathrm{d}_t\eta^{n+1}\right\|^2\right)$$

$$+ \sum_{n=0}^{N-1}\frac{\kappa}{\mu w_n}\left\|R_\alpha\right\|^2 + C\sum_{n=0}^{N-1}\tau\mu^2\left\|\frac{\partial^2}{\partial x\partial y}\left(\mathrm{d}_t\eta^{n+1} - \mathrm{d}_t u^{n+1}\right)\right\|^2. \tag{31}$$

Using the triangle inequality and (31), we get

$$\left\|U^N - u^N\right\|^2 \le C\left(\left\|\xi^N\right\|^2 + \left\|\eta^N\right\|^2\right)$$

$$\le C\left(\left\|\xi^0\right\|^2 + \mu^2 \left\|\frac{\partial^2 \xi^0}{\partial x \partial y}\right\|^2\right) + C\left\|\eta^N\right\|^2 + C\sum_{n=0}^{N-1}\tau\left(\left\|\eta^n\right\|^2 + \left\|d_t\eta^{n+1}\right\|^2\right) \quad (32)$$

$$+ \sum_{n=0}^{N-1}\frac{\kappa}{\mu w_n}\left\|R_\alpha\right\|^2 + C\sum_{n=0}^{N-1}\tau\mu^2\left\|\frac{\partial^2}{\partial x \partial y}\left(d_t\eta^{n+1} - d_t u^{n+1}\right)\right\|^2.$$

It remains to estimate terms on the right-hand side of (32). Firstly, by Lemma 3, we can conclude

$$\left\|\eta^N\right\|^2 \le \left\|\eta\right\|^2_{L^\infty(L^2)} \le Ch^{2r}\left\|u\right\|^2_{L^\infty(H^r)}, \quad (33)$$

$$\sum_{n=0}^{N-1}\tau\left\|\eta^n\right\|^2 \le N\tau\left\|\eta\right\|^2_{L^\infty(L^2)} \le Ch^{2r}\left\|u\right\|^2_{L^\infty(H^r)}. \quad (34)$$

By (10) and $\mu = \tau^\alpha / \Gamma(\alpha+1)$, we easily get

$$\sum_{n=0}^{N-1}\frac{\kappa}{\mu w_n}\left\|R_\alpha\right\|^2 \le C\sum_{n=0}^{N-1}w_n\tau^{2+\alpha} = CN^\alpha\tau^{2+\alpha} = CT^\alpha\tau^2. \quad (35)$$

Secondly, using calculus equality

$$d_t\eta^{n+1} = \frac{1}{\tau}\int_{t_n}^{t_{n+1}}\eta_t(s)\,ds$$

and Hölder inequality, we have

$$\left\|d_t\eta^{n+1}\right\|^2 \le \frac{1}{\tau}\int_{t_n}^{t_{n+1}}\left\|\eta_t(s)\right\|^2\,ds.$$

Further, combined with Lemma 3, we can write

$$\sum_{n=0}^{N-1}\tau\left\|d_t\eta^{n+1}\right\|^2 \le \sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\left\|\eta_t(s)\right\|^2\,ds = \int_0^T\left\|\eta_t(s)\right\|^2\,ds$$

$$= \left\|\eta_t\right\|^2_{L^2(L^2)} \le Ch^{2r}\left\|u_t\right\|^2_{L^2(H^r)}. \quad (36)$$

Since

$$\frac{\partial^2}{\partial x \partial y}\left(d_t\eta^{n+1}\right) = d_t\left(\frac{\partial^2}{\partial x \partial y}\eta^{n+1}\right),$$

similar as (36), we can prove

$$\sum_{n=0}^{N-1}\tau\left\|\frac{\partial^2}{\partial x \partial y}\left(d_t\eta^{n+1}\right)\right\|^2 \le \left\|\frac{\partial^2\eta_t}{\partial x \partial y}\right\|^2_{L^2(L^2)}.$$

Additionally, according to Lemma 3 and Lemma 4, we have

$$\left\|\frac{\partial^2\eta_t}{\partial x \partial y}\right\|^2_{L^2(L^2)} \le Ch^{2(r-2)}\| u_t \|^2_{L^2(H^r)}.$$

As a result, we obtain

$$\sum_{n=0}^{N-1} \tau \mu^2 \left\| \frac{\partial^2}{\partial x \partial y} \left(\mathrm{d}_t \eta^{n+1} \right) \right\|^2 \leq \mu^2 \left\| \frac{\partial^2 \eta_t}{\partial x \partial y} \right\|_{L^2(L^2)}^2 \leq C h^{2(r-2)} \tau^{2\alpha} \| u_t \|_{L^2(H^r)}^2. \tag{37}$$

Similar as (37), we can write

$$\sum_{n=0}^{N-1} \tau \mu^2 \left\| \frac{\partial^2}{\partial x \partial y} \left(\mathrm{d}_t u^{n+1} \right) \right\|^2 \leq \mu^2 \left\| \frac{\partial^2 u_t}{\partial x \partial y} \right\|_{L^2(L^2)}^2 \leq C \tau^{2\alpha} \| u_{xyt} \|_{L^2(L^2)}^2. \tag{38}$$

Combining (33)-(38), and recalling $r \geq 2$ gives

$$\max_{0 \leq n \leq N} \| U^n - u^n \| \leq C \left(\tau^\alpha + h^r \right),$$

provided U^0 is chosen so that $\| \xi^0 \|$ and $\tau^\alpha \| \partial^2 \xi^0 / \partial x \partial y \|$ are $\mathcal{O}(h^r)$. This completes the proof.

Remark 2. Although in our theoretic analysis, we only obtain $O(\tau^\alpha)$ accuracy in temporal error, numerical experiments in Section 6 demonstrate that temporal error is $O(\tau)$ if $1/2 < \alpha < 1$. Checking the above analysis process, we will find the error estimates in (37) and (38) are obstacles to obtain $O(\tau)$ temporal error theoretically. This suggests we may take alternative analysis techniques to improve the error estimates in (37) and (38).

Note that we may choose the initial approximation as $U^0 = W^0$, where W^0 from (18),

$$\left(A \nabla \left(\phi - W^0 \right), \nabla V \right) + \kappa \left(\phi - W, V \right) = 0, \quad \forall V \in \mathcal{M}_{h,r}.$$

This involves an elliptic problem to be solved. With this choice,

$$\| \xi^0 \| = \| \nabla \xi^0 \| = \| \partial^2 \xi^0 / \partial x \partial y \| = 0.$$

In practical computations, it is often sufficient to take U^0 as interpolants of ϕ in $\mathcal{M}_{h,r}$.

5. Matrix Form of ADI FEM

Equations (14) define the ADI finite element method in inner product form. To describe the algebraic problem to which these equations lead, suppose $\mathcal{M}_{h,r} = \mathcal{M}_{h,r}^x \otimes \mathcal{M}_{h,r}^y$, where $\mathcal{M}_{h,r}^x$ and $\mathcal{M}_{h,r}^y$ are finite-dimensional subspaces of $H_0^1(I)$, and let $\{ \varphi_i \theta_j \}_{i=1, j=1}^{N_x, N_y}$ be a tensor product basis for $\mathcal{M}_{h,r}$, where $\{ \varphi_i \}_{i=1}^{N_x}$ and $\{ \theta_j \}_{j=1}^{N_y}$ are bases for the subspaces $\mathcal{M}_{h,r}^x$ and $\mathcal{M}_{h,r}^y$, respectively. Let

$$U^n(x, y) = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \beta_{ij}^{(n)} \varphi_i(x) \theta_j(y),$$

so that

$$E^{n+1}(x, y) = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \gamma_{ij}^{(n+1)} \varphi_i(x) \theta_j(y), \quad \gamma_{ij}^{(n+1)} = \beta_{ij}^{(n+1)} - \beta_{ij}^{(n)}.$$

For convenience, we denote

$$\lambda = \mu\kappa/(1+\mu\kappa), \text{ and } \lambda_i = \mu\kappa_i/(1+\mu\kappa), \quad i = 1,2.$$

If in (14) we choose $V = \varphi_k\theta_m, k = 1, \cdots, N_x, m = 1, \cdots, N_y$, and then

$$\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\left\{(\varphi_i\theta_j, \varphi_k\theta_m) + \left[\lambda_1(\varphi_i'\theta_j, \varphi_k'\theta_m) + \lambda_2(\kappa_i\theta_j', \varphi_k\theta_m')\right] + \lambda_1\lambda_2(\varphi_i'\theta_j', \varphi_k'\theta_m')\right\}\gamma_{ij}^{(n+1)} = \Psi^n. \quad (39)$$

We define the matrices

$$C_x = \left(\int_I \varphi_i(x)\varphi_j(x)\mathrm{d}x\right)_{N_x \times N_x}, \quad C_y = \left(\int_I \theta_i(y)\theta_j(y)\mathrm{d}y\right)_{N_y \times N_y}$$

$$B_x = \left(\int_I \varphi_i'(x)\varphi_j'(x)\mathrm{d}x\right)_{N_x \times N_x}, \quad B_y = \left(\int_I \theta_i'(y)\theta_j'(y)\mathrm{d}y\right)_{N_y \times N_y}$$

and let

$$\beta^{(p)} = \left[\beta_{11}^{(p)}, \beta_{12}^{(p)}, \cdots, \beta_{1N_y}^{(p)}, \beta_{21}^{(p)}, \cdots, \beta_{N_xN_y}^{(p)}\right]^{\mathrm{T}},$$

with $\gamma^{(p)}$ defined similarly. Then the matrix form of (39) is

$$\left[C_x \otimes C_y + \lambda_1 B_x \otimes C_y + \lambda_2 C_x \otimes B_y + \lambda_1\lambda_2 B_x \otimes B_y\right]\gamma^{(n+1)} = \Psi^{(n)}, \quad (40)$$

where, from (14), the components of $\Psi^{(n)}$ are given by

$$\Psi_{km}^{(n)} = \left\{\left(\lambda C_x \otimes C_y + \lambda_1 B_x \otimes C_y + \lambda_2 C_x \otimes B_y\right)\left[-\beta^{(n)} + \sum_{j=0}^{n-1}(w_j - w_{j+1})\beta^{(n-j)}\right]\right\}_{k,m}$$
$$+ \frac{\tau}{1+\mu\kappa}\left(f(U^n), \varphi_k\theta_m\right),$$

and $\otimes$ denotes the tensor product. We may factor (40) in the form

$$\left[C_x \otimes I_{N_y} + \lambda_1 B_x \otimes I_{N_y}\right]\left[I_{N_x} \otimes C_y + I_{N_x} \otimes \lambda_2 B_y\right]\gamma^{(n+1)} = \Psi^{(n)},$$

which is equivalent to

$$\begin{cases}\left[(C_x + \lambda_1 B_x) \otimes I_{N_y}\right]\hat{\gamma}^{(n+1)} = \Psi^{(n)}, \\ \left[I_{N_x} \otimes (C_y + \lambda_2 B_y)\right]\gamma^{(n+1)} = \hat{\gamma}^{(n+1)},\end{cases}$$

where I_{N_x} and I_{N_y} denote the identity matrices of order N_x and N_y, respectively. Thus we determine $\gamma^{(n+1)}$ by solving two sets of independent one-dimensional problems, first

$$(C_x + \lambda_1 B_x)\hat{\gamma}_m^{(n+1)} = \Psi_m^{(n+1)}, \quad m = 1, 2, \cdots, N_y,$$

in the x-direction, where $\hat{\gamma}_m = \left(\hat{\gamma}_{1m}, \hat{\gamma}_{2m}, \cdots, \hat{\gamma}_{N_x,m}\right)^{\mathrm{T}}$, followed by

$$(C_y + \lambda_2 B_y)\gamma_l^{(n+1)} = \hat{\gamma}_l^{(n+1)}, \quad l = 1, 2, \cdots, N_x,$$

in the y-direction, where $\gamma_l = \left(\gamma_{l1}, \gamma_{l2}, \cdots, \gamma_{l,N_y}\right)^{\mathrm{T}}$. Clearly the computation of each of the vectors $\hat{\gamma}^{(n)}$ and $\gamma^{(n)}$ is highly parallel.

6. Numerical Experiments

In this section, two numerical examples are given to demonstrate the effectiveness and

accuracy of the ADI Galerkin finite element methods. In all numerical examples, we take the linear tensor product basis

$$
\varphi_i(x) = \begin{cases} 0, & x \in [x_0, x_{i-1}], \\ \zeta_{i-1}(x), & x \in [x_{i-1}, x_i], \\ 1 - \zeta_i(x), & x \in [x_i, x_{i+1}], \\ 0, & x \in [x_{i+1}, x_{N_x}], \end{cases}
$$

where $\zeta_i(x) = (x - x_i)/h_i^x, h_i^x = x_{i+1} - x_i$. $\theta_j(y)$ is constructed in a similar way. In all examples, we will take the same spatial step size h in each direction, *i.e.* $h_i^x = h_j^y = h$.

In our numerical simulation, we present the errors in L^2 norm

$$
err(\tau, h) = \|U^N - u^N\|,
$$

and numerical convergence orders are computed by

$$
\gamma_t = \log_2 \left(\frac{err(2\tau, h)}{err(\tau, h)} \right), \quad \gamma_h = \log_2 \left(\frac{err(\tau, 2h)}{err(\tau, h)} \right).
$$

Example 1. Consider the following problem

$$
\frac{\partial u(x, y, t)}{\partial t} = {}_0D_t^{1-\alpha} \left[\frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2} - u(x, y, t) \right] + f(u, x, y, t),
$$

$$
(x, y, t) \in \Omega \times (0, T]
$$

where

$$
f(u, x, y, t) = -u^2 + \left(\frac{6}{\Gamma(2 + \alpha)} t^{1+\alpha} + 2t \right) \sin x \sin y + t^4 \sin^2 x \sin^2 y,
$$

with initial and boundary conditions

$$
u(x, y, t) = 0, \quad (x, y) \in \partial\Omega, 0 < t \le T,
$$

$$
u(x, y, 0) = 0, \quad (x, y) \in \Omega,
$$

where $\Omega = (0, \pi) \times (0, \pi)$, $T = 1$. The exact solution of this equation is

$$
u(x, y, t) = t^2 \sin x \sin y.
$$

Table 1 shows the L^2 norm errors and the temporal convergence orders for $\alpha = 0.6$, 0.7, 0.8, 0.9 with fixed $h = \pi/64$ using ADI Galerkin finite element scheme (14). Computational results in Table 1 illustrate that our ADI Galerkin finite element scheme has first order accuracy in time, which is better than our predicted α order in Theorem 2.

To investigate the necessary of the correction term (16) in ADI scheme (17) if $0 < \alpha < \frac{1}{2}$, we have compared the two ADI Galerkin finite element scheme (14) and (17) for $\alpha = 0.1, 0.2, 0.3, 0.4$, with fixed $h = \pi/64$. For saving space, we only present the comparison results of $\alpha = 0.1$ in Table 2, the others cases are similar. The first three columns of Table 2 show the L^2 norm errors and its corresponding temporal convergence orders using ADI scheme (14). Computational results demonstrate the temporal

Table 1. L^2 errors and temporal convergence orders, fixing $h = \pi/64$.

τ	$\alpha = 0.6$		$\alpha = 0.7$		$\alpha = 0.8$		$\alpha = 0.9$	
	$err(\tau, h)$	γ_t	$err(\tau, h)$	γ_t	$err(\tau, h)$	γ_t	$err(\tau, h)$	γ_t
1/8	7.54e−02	--	8.59e−02	--	9.52e−02	--	1.03e−01	--
1/16	3.71e−02	1.02	4.45e−02	0.95	5.00e−02	0.93	5.43e−02	0.92
1/32	1.83e−02	1.02	2.27e−02	0.97	2.56e−02	0.97	2.76e−02	0.98
1/64	8.89e−03	1.04	1.13e−02	1.01	1.27e−02	1.01	1.36e−02	1.02
1/128	4.13e−03	1.11	5.37e−03	1.07	6.02e−03	1.08	6.44e−03	1.08
1/256	1.70e−03	1.28	2.31e−03	1.22	2.61e−03	1.21	2.81e−03	1.20

Table 2. Comparison between ADI scheme (14) and (17) for $\alpha = 0.1$, $h = \pi/64$.

ADI Scheme (14)			ADI Scheme (17)		
τ	$err(\tau, h)$	γ_t	τ	$err(\tau, h)$	γ_t
1/64	8.48e−02	--	1/16	6.96e−02	--
1/128	8.45e−02	0.01	1/32	3.37e−02	1.05
1/256	7.99e−02	0.08	1/64	1.61e−02	1.07
1/512	7.37e−02	0.12	1/128	7.43e−03	1.12
1/1024	6.71e−02	0.14	1/256	3.19e−03	1.22

convergence order is 0.1, which also suggest our theoretical order α proved in Theorem 2 is optimal. The last three columns of **Table 2** display the L^2 norm errors and its corresponding temporal convergence orders using ADI scheme (17). As we can see, the experiment convergence order is approximately one now. It indicates that the correction term (16) in ADI scheme (17) is beneficial to keep the temporal accuracy. In a word, **Table 2** shows ADI scheme (17) has better numerical performance than ADI scheme (14) if $0 < \alpha < \dfrac{1}{2}$.

Table 3 shows the L^2 norm errors and the spatial convergence orders for $\alpha = 0.6$, 0.7, 0.8, 0.9 with fixed $\tau = 1/5000$ using ADI Galerkin finite element scheme (14). The numerical results in **Table 3** suggest our ADI Galerkin finite element scheme gets second order accuracy in space, which is consistent with our theoretical result in Theorem 2.

Example 2. Consider the following problem

$$\frac{\partial u(x,y,t)}{\partial t} = {}_0D_t^{1-\alpha}\left[\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} - u(x,y,t)\right] + f(u,x,y,t),$$
$$(x,y,t) \in \Omega \times (0,T]$$

where

$$f(u,x,y,t) = -u^3 + \left(\frac{2}{\Gamma(2+\alpha)}t^{1+\alpha} + 2t\right)xy(1-x)(1-y) + t^6 x^3 y^3 (1-x)^3 (1-y)^3,$$

with initial and boundary conditions

$$u(x, y, t) = 0, \quad (x, y) \in \partial\Omega, 0 < t \leq T,$$

$$u(x, y, 0) = 0, \quad (x, y) \in \Omega,$$

where $\Omega = (0, \pi) \times (0, \pi)$, $T = 1$. The exact solution of this equation is

$$u(x, y, t) = t^2 x (\pi - x) y (\pi - y).$$

The nonlinearity of Example 2 is stronger than Example 1. In the following simulation, we have to take much smaller temporal step τ than Example 1 to obtain good numerical performance. **Table 4** shows the L^2 norm errors and the temporal convergence orders for $\alpha = 0.6, 0.7, 0.8, 0.9$ with fixed $h = \pi/64$ using ADI Galerkin finite element scheme (14). Computational results in **Table 4** illustrate that our ADI Galerkin finite element scheme is first order temporal accuracy again, if $1/2 < \alpha < 1$.

Table 5 displays the comparison between the two ADI Galerkin finite element scheme (14) and (17) for $\alpha = 0.2$, with fixed $h = \pi/128$. From **Table 5**, we can conclude that: 1) The first three columns show the order of temporal accuracy of ADI scheme (14) is 0.2 for $\alpha = 0.2$, confirming our theoretical result in Theorem 2 again; 2) The last three columns illustrate the ADI scheme (17) has first order temporal accuracy, indicating the essentiality of the correction term (16) to keep temporal accuracy once again; 3) To obtain the same level of temporal truncation error, ADI scheme (17) can take much smaller time step τ than ADI scheme (14) due to its higher accuracy. It also suggests that high order of temporal accuracy is important to time-dependent nonlinear problem.

Table 3. L^2 errors and spatial convergence orders, fixing $\tau = 1/5000$.

h	$\alpha = 0.6$		$\alpha = 0.7$		$\alpha = 0.8$		$\alpha = 0.9$	
	$err(\tau, h)$	γ_h	$err(\tau, h)$	γ_h	$err(\tau, h)$	γ_h	$err(\tau, h)$	γ_h
$\pi/8$	5.63e−02	--	5.60e−02	--	5.57e−02	--	5.54e−02	--
$\pi/16$	1.41e−02	2.00	1.40e−02	2.00	1.39e−02	2.00	1.38e−02	2.01
$\pi/32$	3.33e−03	2.04	3.39e−03	2.05	3.36e−03	2.05	3.33e−03	2.05
$\pi/64$	7.50e−04	2.19	7.21e−04	2.23	7.06e−04	2.25	6.93e−04	2.26

Table 4. L^2 errors and temporal convergence orders, fixing $h = \pi/64$.

τ	$\alpha = 0.6$		$\alpha = 0.7$		$\alpha = 0.8$		$\alpha = 0.9$	
	$err(\tau, h)$	γ_t	$err(\tau, h)$	γ_t	$err(\tau, h)$	γ_t	$err(\tau, h)$	γ_t
1/32	5.78e−01	--	5.84e−01	--	5.89e−01	--	5.93e−01	--
1/64	2.83e−01	1.03	2.86e−01	1.03	2.89e−01	1.03	2.91e−01	1.03
1/128	1.38e−01	1.04	1.40e−01	1.03	1.41e−01	1.04	1.42e−01	1.04
1/256	6.62e−02	1.06	6.71e−02	1.06	6.77e−02	1.06	6.81e−02	1.06
1/512	3.06e−02	1.11	3.10e−02	1.11	3.13e−02	1.11	3.15e−02	1.11

Table 5. Comparison between ADI scheme (14) and (17) for $\alpha = 0.2$, $h = \pi/128$.

	ADI Scheme (14)			ADI Scheme (17)	
τ	$err(\tau, h)$	γ_t	τ	$err(\tau, h)$	γ_t
1/512	4.12e−02	--	1/128	1.42e−01	--
1/1024	3.57e−02	0.21	1/256	6.99e−02	1.02
1/2048	3.08e−02	0.21	1/512	3.42e−02	1.03
1/4096	2.58e−02	0.26	1/1024	1.64e−02	1.06

Table 6. L^2 errors and spatial convergence orders, fixing $\tau = 1/5000$.

h	$\alpha = 0.6$		$\alpha = 0.7$		$\alpha = 0.8$		$\alpha = 0.9$	
	$err(\tau, h)$	γ_h	$err(\tau, h)$	γ_h	$err(\tau, h)$	γ_h	$err(\tau, h)$	γ_h
$\pi/8$	3.85e−01	--	3.82e−01	--	3.79e−01	--	3.76e−01	--
$\pi/16$	9.93e−02	1.95	9.85e−02	1.96	9.77e−02	1.96	9.70e−02	1.95
$\pi/32$	2.35e−02	2.08	2.32e−02	2.09	2.30e−02	2.09	2.28e−02	2.09
$\pi/64$	4.29e−03	2.45	4.20e−03	2.47	4.13e−03	2.48	4.06e−03	2.49

Table 6 shows the L^2 norm errors and the spatial convergence orders for α = 0.6, 0.7, 0.8, 0.9 with fixed $\tau = 1/5000$ using ADI Galerkin finite element scheme (14). It can be shown from Table 6 that the numerical results are in accordance with the theoretical analysis in Theorem 2.

7. Conclusion

In this work, we proposed an alternating direction Galerkin finite element method for 2D nonlinear time fractional reaction sub-diffusion equation in the Riemann-Liouville type. The stability and convergence of the method are proved for $\alpha \in (0,1)$. Numerical results show that our error estimate is optimal. Like the other ADI schemes [16] [17], the temporal accuracy of our ADI Galerkin finite element method becomes very low if $\alpha \in (0,1/2)$. A remedy is introduced by adding a correction term (15) to keep temporal accuracy.

Acknowledgements

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