Open Journal of Mathematical Sciences

Article

Congruence properties of indices of triangular numbers multiple of other triangular numbers

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Academic Editor: Tuncer Acar
Received: 5 March 2021; Accepted: 10 June 2021; Published: 14 July 2021.

Abstract: For any non-square integer multiplier k, there is an infinity of triangular numbers multiple of other triangular numbers. We analyze the congruence properties of indices ξ of triangular numbers multiple of triangular numbers. Remainders in congruence relations ξ modulo k come always in pairs whose sum always equal $(k-1)$, always include 0 and $(k-1)$, and only 0 and $(k-1)$ if k is prime, or an odd power of a prime, or an even square plus one or an odd square minus one or minus two. If the multiplier k is twice the triangular number of n, the set of remainders includes also n and (n^2-1) and if k has integer factors, the set of remainders include multiples of a factor following certain rules. Algebraic expressions are found for remainders in function of k and its factors, with several exceptions. This approach eliminates those ξ values not providing solutions.

Keywords: Triangular numbers; Multiple of triangular numbers; Recurrent relations; Congruence properties.

MSC: Primary 11A25; Secondary 11D09.

1. Introduction

Triangular numbers $T_t = \frac{t(t+1)}{2}$ are one of the figurate numbers enjoying many properties; see, e.g., [1,2] for relations and formulas. Triangular numbers T_ξ that are multiples of other triangular number T_t

$$T_\xi = kT_t \tag{1}$$

are investigated. Only solutions for $k > 1$ are considered as the cases $k = 0$ and $k = 1$ yield respectively $\xi = 0$ and $\xi = t, \forall t$. Accounts of previous attempts to characterize these triangular numbers multiple of other triangular numbers can be found in [3–9]. Recently, Pletser [9] showed that, for non-square integer values of k, there are infinitely many solutions that can be represented simply by recurrent relations of the four variables t, ξ, Tt and T_ξ, involving a rank r and parameters κ and γ, which are respectively the sum and the product of the $(r-1)^{\text{th}}$ and the r^{th} values of t. The rank r is being defined as the number of successive values of t solutions of (1) such that their successive ratios are slowly decreasing without jumps.

In this paper, we present a method based on the congruent properties of $\xi \pmod k$, searching for expressions of the remainders in function of k or its factors. This approach accelerates the numerical search of the values of t_n and ξ_n that solve (1), as it eliminates values of ξ that are known not to provide solutions to (1). The gain is typically in the order of k/v where v is the number of remainders, which is usually such that $v \ll k$.

2. Rank and Recurrent Equations

Sequences of solutions of (1) are known for $k = 2, 3, 5, 6, 7, 8, 10$ and are listed in the Online Encyclopedia of Integer Sequences (OEIS) [10], with references given in Table 1.

Among all solutions, $t = 0$ is always a first solution of (1) for all non-square integer value of k, yielding $\xi = 0$.

Let's consider the two cases of $k = 2$ and $k = 7$ yielding the successive solution pairs as shown in Table 2. We indicate also the ratios t_n/t_{n-1} for both cases and t_n/t_{n-2} for $k = 7$. It is seen that for $k = 2$, the ratio t_n/t_{n-1} varies between close values, from 7 down to 5.829, while for $k = 7$, the ratio t_n/t_{n-1} alternates between values 2.5 ... 2.216 and 7.8 ... 7.23, while the ratio t_n/t_{n-2} decreases regularly from 19.5 to 16.023

Table 1. OEIS [10] references of sequences of integer solutions of (1) for $k = 2, 3, 5, 6, 7, 8, 10$

k	2	3	5	6	7	8	10
t	A053141	A061278	A077259	A077288	A077398	A336623	A341893
ξ	A001652	A001571	A077262	A077291	A077401	A336625	A341895
T_t	A075528	A076139	A077260	A077289	A077399	A336624	A068085
T_ξ	A029549	A076140	A077261	A077290	A077400	A336626	–

Table 2. Solutions of (1) for $k = 2, 7$

n	$k=2$			$k=7$			
	t_n	ξ_n	$\frac{t_n}{t_{n-1}}$	t_n	ξ_n	$\frac{t_n}{t_{n-1}}$	$\frac{t_n}{t_{n-2}}$
0	0	0		0	0		
1	2	3	–	2	6	–	–
2	14	20	7	5	14	2.5	–
3	84	119	6	39	104	7.8	19.5
4	492	696	5.857	87	231	2.231	17.4
5	2870	4059	5.833	629	1665	7.230	16.128
6	16730	23660	5.829	1394	3689	2.216	16.023

(corresponding approximately to the product of the alternating values of the ratio t_n/t_{n-1}). We call rank r the integer value such that t_n/t_{n-r} is approximately constant or, better, decreases regularly without jumps (a more precise definition is given further). So, here, the case $k = 2$ has rank $r = 1$ and the case $k = 7$ has rank $r = 2$.

In [9], we showed that the rank r is the index of t_r and ξ_r solutions of (1) such that

$$\kappa = t_r + t_{r-1} = \xi_r - \xi_{r-1} - 1, \tag{2}$$

and that the ratio t_{2r}/t_r, corrected by the ratio t_{r-1}/t_r, is equal to a constant $2\kappa + 3$

$$\frac{t_{2r} - t_{r-1}}{t_r} = 2\kappa + 3. \tag{3}$$

For example, for $k = 7$ and $r = 2$, (2) and (3) yield respectively, $\kappa = 7$ and $2\kappa + 3 = 17$.

Four recurrent equations for t_n, ξ_n, T_{t_n} and T_{ξ_n} are given in [9] for each non-square integer value of k

$$t_n = 2(\kappa+1)\, t_{n-r} - t_{n-2r} + \kappa, \tag{4}$$

$$\xi_n = 2(\kappa+1)\, \xi_{n-r} - \xi_{n-2r} + \kappa, \tag{5}$$

$$T_{t_n} = \left(4(\kappa+1)^2 - 2\right) T_{t_{n-r}} - T_{t_{n-2r}} + (T_\kappa - \gamma), \tag{6}$$

$$T_{\xi_n} = \left(4(\kappa+1)^2 - 2\right) T_{\xi_{n-r}} - T_{\xi_{n-2r}} + k(T_\kappa - \gamma), \tag{7}$$

where coefficients are functions of two constants κ and γ, respectively the sum κ and the product $\gamma = t_{r-1}t_r$ of the first two sequential values of t_r and t_{r-1}. Note that the first three relations (4) to (6) are independent of the value of k.

3. Congruence of ξ modulo k

We use the following notations: for $A, B, C \in \mathbb{Z}, B < C, C > 1$, $A \equiv B \pmod C$ means that $\exists D \in \mathbb{Z}$ such that $A = DC + B$, where B and C are called respectively the remainder and the modulus. To search numerically for the values of t_n and ξ_n that solve (1), one can use the congruent properties of $\xi \pmod k$ given in the following propositions. In other words, we search in the following propositions for expressions of the remainders in function of k or its factors.

Proposition 1. *For $\forall s, k \in \mathbb{Z}^+$, k non-square, $\exists\ \xi, \mu, v, i, j \in \mathbb{Z}^+$, such that if ξ_i are solutions of (1), then for $\xi_i \equiv \mu_j \pmod k$ with $1 \le j \le v$, the number v of remainders is always even, $v \equiv 0 \pmod 2$, the remainders come in pairs* *whose sum is always equal to $(k-1)$, and the sum of all remainders is always equal to the product of $(k-1)$ and the number of remainder pairs, $\sum_{j=1}^{v} \mu_j = (k-1)\,v/2$.*

Proof. Let $s, i, j, k, \xi, \mu, v, \alpha, \beta \in \mathbb{Z}^+$, k non-square, and ξ_i solutions of (1). Rewriting (1) as $T_{t_i} = T_{\xi_i}/k$, for T_{t_i} to be integer, k must divide exactly $T_{\xi_i} = \xi_i(\xi_i+1)/2$, i.e., among all possibilities, k divides either ξ_i or (ξ_i+1), yielding two possible solutions $\xi_i \equiv 0 \pmod k$ or $\xi_i \equiv -1 \pmod k$, i.e., $v = 2$ and the set of μ_j includes $\{0, (k-1)\}$. This means that ξ_i are always congruent to either 0 or $(k-1)$ modulo k for all non-square values of k.

Furthermore, if some ξ_i are congruent to α modulo k, then other ξ_i are also congruent to β modulo k with $\beta = (k-\alpha-1)$. As $\xi_i \equiv \alpha \pmod k$, then $\xi_i(\xi_i+1)/2 \equiv (\alpha(\alpha+1)/2) \pmod k$ and replacing α by $\alpha = (k-\beta-1)$ yields $(\alpha(\alpha+1)/2) = ((k-\beta-1)(k-\beta)/2)$, giving $\xi_i(\xi_i+1)/2 \equiv ((k-\beta-1)(k-\beta)/2) \pmod k \equiv (\beta(\beta+1)/2) \pmod k$. In this case, $v = 4$ and the set of μ_j includes, but not necessarily limits to, $\{0, \alpha, (k-\alpha-1), (k-1)\}$. □

Note that in some cases, $v > 4$, as for $k = 66, 70, 78, 105, ..., v = 8$. However, in some other cases, $v = 2$ only and the set of μ_j contains only $\{0, (k-1)\}$, as shown in the next proposition. In this proposition, several rules (R) are given constraining the congruence characteristics of ξ_i.

Proposition 2. *For $\forall\ s, k, \alpha, n \in \mathbb{Z}^+$, k non-square, $\alpha > 1$, $\exists\ \xi, \mu, v, i \in \mathbb{Z}^+$, such that if ξ_i are solutions of (1), then ξ_i are always only congruent to 0 and $(k-1)$ modulo k, and $v = 2$ if either*

(R1) k is prime, or
(R2) $k = \alpha^n$ with α prime and n odd, or
(R3) $k = s^2+1$ with s even, or
(R4) $k = s'^2-1$ or (R5) $k = s'^2-2$ with s' odd.

Proof. Let $s, s', k, \alpha > 1, n, i, \xi, \mu, v \in \mathbb{Z}^+$, k non-square, and ξ_i are solutions of (1).
(R1)+(R2): If k is prime or if $k = \alpha^n$ (with α prime and n odd as k is non-square), then, in both cases, k can only divide either ξ_i or (ξ_i+1), yielding the two congruences $\xi_i \equiv 0 \pmod k$ and $\xi_i \equiv -1 \pmod k$.
(R3): If $k = s^2+1$ with s even, the rank r is always $r = 2$ [11], and the only two sets of solutions are

$$(t_1, \xi_1) = \left(s(s-1), \left(s^2+1\right)(s-1)\right) \tag{8}$$
$$(t_2, \xi_2) = \left(s(s+1), \left(s^2+1\right)(s+1)-1\right) \tag{9}$$

as can be easily shown. For t_1, forming

$$\begin{aligned} kT_{t_1} &= \frac{1}{2}\left(s^2+1\right)(s(s-1))(s(s-1)+1) \\ &= \frac{1}{2}\left[\left(s^2+1\right)(s-1)\right]\left[\left(s^2+1\right)(s-1)+1\right] = T_{\xi_1} \end{aligned}$$

which is the triangular number of ξ_1. One obtains similarly ξ_2 from t_2. These two relations (8) and (9) show respectively that ξ_1 is congruent to 0 modulo k and ξ_2 is congruent to $(k-1)$ modulo k.
(R4): For $k = s'^2-1$ with s' odd, the rank $r = 2$ [11], and the only two sets of solutions are

$$(t_1, \xi_1) = \left((s'-1)s'-1, \left(s'^2-1\right)(s'-1)-1\right) \tag{10}$$
$$(t_2, \xi_2) = \left((s'-1)(s'+2)+1, \left(s'^2-1\right)(s'+1)\right) \tag{11}$$

as can be easily demonstrated as above. These two relations (10) and (11) show that ξ_1 and ξ_2 are congruent respectively to $(k-1)$ and 0 modulo k.
(R5): For $k = s'^2-2$ with s' odd, the rank $r = 2$ [11], and the only two sets of solutions are

$$(t_1, \xi_1) = \left(\frac{1}{2}(s'-2)(s'+1), \frac{1}{2}\left(s'^2-2\right)(s'-1)-1\right) \tag{12}$$
$$(t_2, \xi_2) = \left(\frac{s'}{2}(s'+1)-1, \frac{1}{2}\left(s'^2-2\right)(s'+1)\right) \tag{13}$$

Table 3. Combination of parameters m and ν for $2 \leq n \leq 12$

m ↘ n	ν = 1	2	3	4	5	6	7	8	9	10	11
2	1_										
3	1_	1									
4	1_	/	1								
5	1_	2	2_	1							
6	1_	/	/	/	1						
7	1_	3	2	2_	3_	1					
8	1_	/	3_	/	3	/	1				
9	1_	4	/	2	2_	/	4_	1			
10	1_	/	3	/	5_	/	3_	/	1		
11	1_	5	4_	3_	2	2_	3	4	5_	1	
12	1_	/	/	/	3	/	4_	/	/	/	1

as can easily be shown as above. These two relations (12) and (13) show that ξ_1 and ξ_2 are congruent respectively to $(k-1)$ and 0 modulo k. □

There are other cases of interest as shown in the next two Propositions:

Proposition 3. *For* $\forall\ n \in \mathbb{Z}^+$, $\exists\ k, \xi, \mu < k, i, j \in \mathbb{Z}^+$, *$k$ non-square, such that if ξ_i are solutions of (1) with $\xi_i \equiv \mu_j \pmod{k}$, and (R6) if k is twice a triangular number* $k = n(n+1) = 2T_n$, *then the set of μ_j includes* $\{0, n, (n^2-1), (k-1)\}$, *with* $1 \leq j \leq v$.

Proof. Let $n, k, \xi, \mu < k, i, j \in \mathbb{Z}^+$, k non-square, and ξ_i solutions of (1). Let $\xi_i \equiv \mu_j \pmod{k}$ with $1 \leq j \leq v$. As the ratio $\xi_i(\xi_i+1)/k$ must be integer, $\xi_i(\xi_i+1) \equiv 0 \pmod{k}$ or $\mu_j(\mu_j+1) \equiv 0 \pmod{n(n+1)}$ which is obviously satisfied if $\mu_j = n$ or $\mu_j = (n^2-1)$. □

Finally, this last proposition gives a general expression of the congruence $\xi_i \pmod{k}$ for most cases to find the remainders μ_j other than 0 and $(k-1)$.

Proposition 4. *For* $\forall n > 1 \in \mathbb{Z}^+$, $\exists k, f, \xi, \nu < n < k, \mu < k, m < n, i, j \in \mathbb{Z}^+$, *$k$ non-square, let ξ_i be solutions of (1) with $\xi_i \equiv \mu_j \pmod{k}$, let f be a factor of k such that $f = k/n$ with $f \equiv \nu \pmod{n}$ and $k \equiv \nu n \pmod{n^2}$, then the set of μ_j includes either $\{0, mf, ((n-m)f-1), (k-1)\}$ or $\{0, (mf-1), (n-m)f, (k-1)\}$, where m is an integer multiplier of f in the congruence relation and such that $m < n/2$ or $m < (n+1)/2$ for n being even or odd respectively, and* $1 \leq j \leq v$.

Proof. Let $n > 1, k, f, \xi, \mu < k, m < n, i, j < n < k \in \mathbb{Z}^+$, k non-square, and ξ_i a solution of (1). Let $\xi_i \equiv \mu_j \pmod{k}$ with $1 \leq j \leq v$. As the ratio $\xi_i(\xi_i+1)/k$ must be integer, $\xi_i(\xi_i+1) \equiv 0 \pmod{k}$ or $\mu_j(\mu_k+1) \equiv 0 \pmod{fn}$. For a proper choice of the factor f of k, let μ_j be a multiple of f, $\mu_j = mf$, then $m(mf+1) \equiv 0 \pmod{n}$. As $f \equiv \nu \pmod{n}$, one has

$$m(m\nu+1) \equiv 0 \pmod{n}. \tag{14}$$

Let now (μ_j+1) be a multiple of f, $\mu_j + 1 = mf$, then $m(mf-1) \equiv 0 \pmod{n}$ or

$$m(m\nu-1) \equiv 0 \pmod{n}. \tag{15}$$

An appropriate combination of integer parameters m and ν guarantees that (14) and (15) are satisfied. Proposition 1 yields the other remainder value as $mf + (n-m)f - 1 = k-1$ and $(mf-1) + (n-m)f = k-1$. □

The appropriate combinations of integer parameters m and ν are given in Table 3 for $2 \leq n \leq 12$. The sign $-$ in subscript corresponds to the remainder $(mf-1)$; the sign / indicates an absence of combination.

One deduces from Table 3 the following simple rules:

1) $\forall n \in \mathbb{Z}^+$, only those values of ν that are co-prime with n must be kept, all other combinations (indicated by / in Table 3) must be discarded as they correspond to combinations with smaller values of n and ν;

for n even, this means that all even values of ν must be discarded. For example, $\nu = 2$ and $n = 4$ are not co-prime and their combination corresponds to $\nu = 1$ and $n = 2$.

2) For $\nu = 1$ and $\nu = n - 1$, all values of m are $m = 1$ with respectively the remainders $(mf - 1)$ and mf.
3) For $\forall n, i \in \mathbb{Z}^+$, n odd, $2 \leq i \leq (n-1)/2$, and for $\nu = (n - (2i-3))/2$ and $\nu = (n + (2i-3))/2$, all the values of m are $m = i$.
4) For $\forall n \in \mathbb{Z}^+$, n odd, and for $\nu = 2$ and $\nu = n - 2$, the remainders are respectively mf and $(mf - 1)$.
5) For $\forall n, i \in \mathbb{Z}^+$, n even, $2 \leq i \leq n/2$, and for $\nu = (n - (2i-3))/2$ and $\nu = (n + (2i-3))/2$, all the values of m are $m = i$.

Expressions of μ_i are given in Table 4 for $2 \leq n \leq 12$ (with codes E$n\nu$). For example, for $k \equiv 12\nu \pmod{12^2}$ and $\nu = 5$ (code E125), i.e. $k = 60, 204, 348, \ldots$, $\xi_i \equiv \mu_j \pmod{k}$ with the set of remainders μ_j including $\{0, mf, ((n-m)f - 1), (k-1)\}$ with $m = 3$ (see Table 3) and $f = k/12 = 5, 17, 29\ldots$respectively.

Table 4. Expressions of μ_j for $2 \leq n \leq 12$

n	ν	m	$k \equiv$	f	μ_j	Code
2	1	1	$2 \pmod 4$	$k/2$	$0, (k/2)-1, k/2, k-1$	E21
3	1	1	$3 \pmod 9$	$k/3$	$0, (k/3)-1, 2k/3, k-1$	E31
	2	1	$6 \pmod 9$		$0, k/3, (2k/3)-1, k-1$	E32
4	1	1	$4 \pmod{16}$	$k/4$	$0, (k/4)-1, 3k/4, k-1$	E41
	3	1	$12 \pmod{16}$		$0, k/4, (3k/4)-1, k-1$	E43
5	1	1	$5 \pmod{25}$	$k/5$	$0, (k/5)-1, 4k/5, k-1$	E51
	2	2	$10 \pmod{25}$		$0, 2k/5, (3k/5)-1, k-1$	E52
	3	2	$15 \pmod{25}$		$0, (2k/5)-1, 3k/5, k-1$	E53
	4	1	$20 \pmod{25}$		$0, k/5, (4k/5)-1, k-1$	E54
6	1	1	$6 \pmod{36}$	$k/6$	$0, (k/6)-1, 5k/6, k-1$	E61
	5	1	$30 \pmod{36}$		$0, k/6, (5k/6)-1, k-1$	E65
7	1	1	$7 \pmod{49}$	$k/7$	$0, (k/7)-1, 6k/7, k-1$	E71
	2	2	$14 \pmod{49}$		$0, 3k/7, (4k/7)-1, k-1$	E72
	3	3	$21 \pmod{49}$		$0, 2k/7, (5k/7)-1, k-1$	E73
	4	3	$28 \pmod{49}$		$0, (2k/7)-1, 5k/7, k-1$	E74
	5	2	$35 \pmod{49}$		$0, (3k/7)-1, 4k/7, k-1$	E75
	6	1	$42 \pmod{49}$		$0, k/7, (6k/7)-1, k-1$	E76
8	1	1	$8 \pmod{64}$	$k/8$	$0, (k/8)-1, 7k/8, k-1$	E81
	3	3	$24 \pmod{64}$		$0, (3k/8)-1, 5k/8, k-1$	E83
	5	3	$40 \pmod{64}$		$0, 3k/8, (5k/8)-1, k-1$	E85
	7	1	$56 \pmod{64}$		$0, k/8, (7k/8)-1, k-1$	E87
9	1	1	$9 \pmod{81}$	$k/9$	$0, (k/9)-1, 8k/9, k-1$	E91
	2	4	$18 \pmod{81}$		$0, 4k/9, (5k/9)-1, k-1$	E92
	4	2	$36 \pmod{81}$		$0, 2k/9, (7k/9)-1, k-1$	E94
	5	2	$45 \pmod{81}$		$0, (2k/9)-1, 7k/9, k-1$	E95
	7	4	$63 \pmod{81}$		$0, (4k/9)-1, 5k/9, k-1$	E97
	8	1	$72 \pmod{81}$		$0, k/9, (8k/9)-1, k-1$	E98
10	1	1	$10 \pmod{100}$	$k/10$	$0, (k/10)-1, 9k/10, k-1$	E101
	3	3	$30 \pmod{100}$		$0, 3k/10, (7k/10)-1, k-1$	E103
	7	3	$70 \pmod{100}$		$0, (3k/10)-1, 7k/10, k-1$	E107
	9	1	$90 \pmod{100}$		$0, k/10, (9k/10)-1, k-1$	E109
11	1	1	$11 \pmod{121}$	$k/11$	$0, (k/11)-1, 10k/11, k-1$	E111
	2	5	$22 \pmod{121}$		$0, 5k/11, (6k/11)-1, k-1$	E112
	3	4	$33 \pmod{121}$		$0, (4k/11)-1, 7k/11, k-1$	E113
	4	3	$44 \pmod{121}$		$0, (3k/11)-1, 8k/11, k-1$	E114
	5	2	$55 \pmod{121}$		$0, 2k/11, (9k/11)-1, k-1$	E115
	6	2	$66 \pmod{121}$		$0, (2k/11)-1, 9k/11, k-1$	E116
	7	3	$77 \pmod{121}$		$0, 3k/11, (8k/11)-1, k-1$	E117
	8	4	$88 \pmod{121}$		$0, 4k/11, (7k/11)-1, k-1$	E118
	9	5	$99 \pmod{121}$		$0, (5k/11)-1, 6k/11, k-1$	E119
	10	1	$110 \pmod{121}$		$0, k/11, (10k/11)-1, k-1$	E1110
12	1	1	$12 \pmod{144}$	$k/12$	$0, (k/12)-1, 11k/12, k-1$	E121
	5	3	$60 \pmod{144}$		$0, 3k/12, (9k/12)-1, k-1$	E125
	7	4	$84 \pmod{144}$		$0, (4k/12)-1, 8k/12, k-1$	E127
	11	1	$132 \pmod{144}$		$0, k/12, (11k/12)-1, k-1$	E1211

Table 5. Values of μ_j for $2 \leq k \leq 120$

k	μ_j	References	k	μ_j	References
2	0,1	R1,R6,E21	63	0,27,35,62	E72,E97
3	0,2	R1,E31	65	0,64	R3
5	0,4	R1,R3,E51	66	0,11,21,32,33,44,54,65	E21+E31+E65+E116
6	0,2,3,5	R6,E21,E32,E61	67	0,66	R1
7	0,6	R1,R5,E71	68	0,16,51,67	E41
8	0,7	R2,R4,E81	69	0,23,45,68	E32
10	0,4,5,9	E21,E52,E101	70	0,14,20,34,35,49,55,69	E21+E54+E73+E107
11	0,10	R1,E111	71	0,70	R1
12	0,3,8,11	R6,E31,E43,E121	72	0,8,63,71	R6,E81,E98
13	0,12	R1	73	0,72	R1
14	0,6,7,13	E21,E72	74	0,73	?
15	0,5,9,14	E32,E53	75	0,24,50,74	E31
17	0,16	R1,R3	76	0,19,56,75	E43
18	0,8,9,17	E21,E92	77	0,21,55,76	E74,E117
19	0,18	R1	78	0,12,26,38,39,51,65,77	E21+E32+E61
20	0,4,15,19	R6,E41,E54	79	0,78	R1,R5
21	0,6,14,20	E31,E73	80	0,79	R4
22	0,10,11,21	E21,E112	82	0,40,41,81	E21
23	0,22	R1,R5	83	0,82	R1
24	0,23	R4	84	0,27,56,83	E31,E127
26	0,12,13,25	E21	85	0,34,50,84	E52
27	0,26	R2	86	0,42,43,85	E21
28	0,7,20,27	E43,E74	87	0,29,57,86	E32
29	0,28	R1	88	0,32,55,87	E83,E118
30	0,5,24,29	R6,E51,E65	89	0,88	R1
31	0,30	R1	90	0,9,80,89	R6,E91,E109
32	0,31	R2	91	0,13,77,90	E75
33	0,11,21,32	E32,E113	92	0,23,68,91	E43
34	0,16,17,33	E21	93	0,30,62,92	E31
35	0,14,20,34	E52,E75	94	0,46,47,93	E21
37	0,36	R1,R3	95	0,19,75,94	E54
38	0,18,19,37	E21	96	0,32,63,95	E32
39	0,12,26,38	E31	97	0,96	R1
40	0,15,24,39	E53,E85	98	0,48,49,97	E21
41	0,40	R1	99	0,44,54,98	E92,E119
42	0,6,35,41	R6,E61,E76	101	0,100	R1,R3
43	0,42	R1	102	0,50,51,102	E21
44	0,11,32,43	E43,E114	103	0,102	R1
45	0,9,35,44	E54,E95	104	0,103	?
46	0,22,23,245	E21	105	0,14,20,35,69,84,90,104	E32+E51+E71
47	0,46	R1,R5	106	0,52,53,105	E21
48	0,47	R4	107	0,106	R1
50	0,24,25,49	E21	108	0,27,80,107	E43
51	0,17,33,50	E32	109	0,108	R1
52	0,12,39,51	E41	110	0,10,99,109	R6,E101,E1110
53	0,52	R1	111	0,36,74,110	E31
54	0,26,27,53	E21	112	0,48,63,111	E72
55	0,10,44,54	E51,E115	113	0,112	R1
56	0,7,48,55	R6,E71,E87	114	0,56,57,113	E21
57	0,18,38,56	E31	115	0,45,69,114	E53
58	0,28,29,57	E21	116	0,28,87,115	E41
59	0,58	R1	117	0,26,90,116	E94
60	0,15,44,59	E43,E125	118	0,58,59,117	E21
61	0,60	R1	119	0,118	R1,R5
62	0,30,31,61	E21	120	0,15,104,119	E87

Values of the remainders μ_j are given in Table 5 for $2 \leq k \leq 120$, with rule (R) and expression (E) codes as references. R and E codes separated by commas imply that all references apply simultaneously to the case; E codes separated by + mean that all expressions apply to the case; some expression references are sometimes missing. One observes that in two cases (for $k = 74$ and 104), expressions could not be found (indicated by question marks).

Table 5 gives correctly the values of the remainder pairs in most of the cases. There are although some exceptions and some values missing.

Among the exceptions to the values given in Table 5, for $n = 2$, remainders values for $k = 30, 42, 74, 90, 110, \ldots$ are different from the theoretical ones in Table 4. Furthermore, for $k = 66, 70, 78, 105, ...$, additional remainders exist. Expressions are missing for $k = 74$ (E21) and 104 (E85). Finally, one observes also that for 16 cases, some Rules or Expressions supersede some other Expressions (indicated by Ra > Exy or Exy > Ezt), as reported in Table 6. For example, Rule 6 supersedes Expression 21 (R6 > E21) for $k = 30, 42, 90, 110$, i.e., $k = 2T_5, 2T_6, 2T_9, 2T_{10}, ...$ and more generally for all $k = 2T_i$ for $i \equiv 1, 2 \pmod 4$.

Table 6. Rules and Expressions superseding other Rules and Expressions

k	
24	R4 > E32; R4 > E83
30	R6 > E21; R6 > E31; R6 > E103; E51 > E103; E65 > E103
42	R6 > E21; R6 > E32
48	R4 > E31
56	R6 > E43
60	E43 > E32; E43 > E52
65	R3 > E53
72	R6 > E43
80	R4 > E51
84	E31 > E41; E31 > E75
90	R6 > E21; R6 > E53
102	E21 > E31; E21 > E65
110	R6 > E21; R6 > E52
114	E21 > E32; E21 > E61
119	R1 > E73; R5 > E73
120	E87 > R4; E87 > E31; E87 > E54

Note that 11 of these 16 values of k are multiples of 6, the others are 2 mod 6 and 5 mod 6 for, respectively three and two cases. One notices as well, that generally, Ra and Exy supersede Ezt with $x < z$ and $t < y$, except for $k = 60$ and 120.

4. Conclusions

We have shown that, for indices ξ of triangular numbers multiples of other triangular numbers, the remainders in the congruence relations of ξ modulo k always come in pairs whose sum always equal $(k-1)$, always include 0 and $(k-1)$, and only 0 and $(k-1)$ if k is prime, or an odd power of a prime, or an even square plus one or an odd square minus one or minus two. If the multiplier k is twice a triangular number of n, the set of remainders includes also n and $\left(n^2-1\right)$ and if k has integer factors, the set of remainders include multiple of a factor following certain rules. Finally, algebraic expressions are found for remainders in function of k and its factors. Several exceptions are noticed as well as reported above and it appears that there are superseding rules between the various rules and expressions.

This approach allows eliminating in numerical searches those $(k-v)$ values of ξ_i that are known not to provide solutions of (1), where v is the even number of remainders. The gain is typically in the order of k/v, with $v \ll k$ for large values of k.

Conflicts of Interest: The author declares no conflict of interest.

Data Availability: Data required for this research is included within this paper.

Funding Information: The author received no financial support for the research authorship and / or publication of this article.

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