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Mickens Iteration Like Method for Approximate Solutions of the Inverse Cubic Truly Nonlinear Oscillator

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Abstract

A new technique of the Mickens iterative method has been presented to obtain approximate analytic solutions of the Inverse Cubic Truly Nonlinear Oscillator. In this paper, we have used Fourier series and utilized truncated terms in each steps of iteration. The method is illustrated by an example and the solutions obtained by this method agree nicely with the exact frequency. Also the solutions give more accurate result than other existing results and the method is convergent.

Keywords: Iterative method; nonlinear oscillators; inverse cubic oscillator; frequency; fourier series.

AMS subject classification: 34A34, 34B99.

1 Introduction

Problem of nonlinear oscillators occupies many researchers. Namely, the nonlinear oscillations occur in many real systems from macro to nano in size, and are the basic or auxiliary motions which follow the main motion. Thus, nonlinear oscillations are evident in many fields of science, not only in physics, mechanics and mathematics but also in electronics, chemistry, biology and astronomy. It has been a research subject of intension focus because most of the oscillatory systems are very often governed by a system of nonlinear differential equation. To solve this type of problems sometimes it is possible to replace a nonlinear differential equation with a related linear equation that approximates the original nonlinear equation closely

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enough to provide useful results. But such linearization is not always possible or feasible. In this situation there are several analytic approaches to find approximate solutions to nonlinear problems.

To deal with problems of nonlinear oscillation systems, the most widely used analytical technique is the perturbation method [1,2] which is, in principle, useful if there exist small parameters in the nonlinear problems. The parameters are analytically expanded into power series of the parameter. The coefficients of the series are found as solutions of a set of linear problems. However, in both science and engineering, there exist many nonlinear problems without small parameters. Even if there exists such a parameter, the analytical solutions given by the perturbation methods have, in most cases, a small validity. Thus, there are many other methods developed for solving nonlinear oscillation systems, including the Krylov-Bogoliubov-Mtropolskii (KBM) method [3-8], the Lindstedt-Poincare method [9,10] and the multiple scales method [11] which is valid even for rather large amplitudes of oscillation. However, it is usually difficult to achieve higher-order analytic approximations by using these methods. Recently, the weighted linearization method, the modified Lindstedt-Poincare method, power series approach and homotopy analysis method [12-15] have been presented for obtaining approximate periods with large amplitude of oscillations. But these methods involve tedious derivations and computations, and they are difficult to implement. Further, they are applicable only to nonlinear equations with odd nonlinearity.

Another one is harmonic balance (HB) method which provides a general technique for calculating approximations to the periodic solutions of differential equations. It corresponds to a truncated Fourier series and allows for the systematic determination of the coefficients to the various harmonics and the angular frequency. HB method which is originated by Mickens [16] and farther work has been done by Wu [17], Gottlieb [18], Hosen [19] and so on for solving the strong nonlinear problems. A new approach, using a rational harmonic balance formulation, was introduced by Belendez et al. [20] they demonstrate the utility of the procedure by applying it to several nonlinear oscillatory systems. The mathematical foundations of harmonic balancing have been investigated by several individuals. The works of Borges et al. [21], Miletta [22] and Bobylev et al. [23] provide overviews to various issues concerning convergence and error bounds for the approximations to the periodic solutions.

Nowadays iteration method is used widely by some authors like Mickens [24-26], Hu and Tang [27] and Haque et al. [28-30] etc. which is valid for small together with large amplitude of oscillation to attain the approximate frequency and the harmonious periodic solution of such nonlinear problems. Mickens [24] provided a general basis for iteration methods as they are currently used in the calculation of approximations to the periodic solutions of nonlinear oscillatory differential equations. A generalization of this work was then given by Lim and Wu [31] and this was followed by an additional extension in Mickens. Actually iteration method is a technique for calculating approximations to the periodic solutions of the truly nonlinear oscillator differential equations which is patented by R.E. Mickens in 1987.

The main purpose of this article is to develop a modification of the iteration technique for the determination of approximate solution and angular frequency of inverse cubic nonlinear oscillator. We compare the result with existing results obtained by various researchers and it is mentioned that our solution measure better results than other existing procedures the method is convergent.

2 The Method

Assume that the nonlinear oscillator

$$F(\ddot{x}, x) = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{1}$$

and further assume that it can be rewritten to the form

$$\ddot{x} + f(\ddot{x}, x) = 0, \tag{2}$$

where over dots denote differentiation with respect to time , t.

We choose the natural frequency Ω of this system. Then adding $\Omega^2 x$ to both sides of Eq. (2), we obtain

$$\ddot{x}+\Omega^2 \mathrm{x}=\Omega^2 \mathrm{x} - f(\ddot{x},x) \equiv G(x,\ddot{x})\,. \tag{3}$$

Now, we formulate the iteration scheme as

$$\ddot{x}_{k+1}+\Omega_k^2 x_{k+1}=G(x_k,\ddot{x}_k);\;\; k=0,1,2,3,............, \tag{4}$$

together with initial condition

$$x_0(t)=A\cos(\Omega_0 t). \tag{5}$$

Hence x_{k+1} satisfies the initial conditions

$$x_{k+1}(0)=A, \quad \dot{x}_{k+1}(0)=0. \tag{6}$$

At each stage of the iteration, Ω_k is determined by the requirement that secular terms [1] should not occur in the full solution of $x_{k+1}(t)$.

The above procedure gives the sequence of solutions: $x_0(t),\, x_1(t),\, x_2(t),\cdots.$

The method can be proceed to any order of approximation; but due to growing algebraic complexity the solution is confined to a lower order usually the second [24].

2.1 Solution procedure

Let us consider the inverse cubic nonlinear oscillator

$$\ddot{x}+x^{-1/3}=0.$$

$$\ddot{x}=-x^{-1/3}. \tag{7}$$

Adding $\Omega^2 x$ on both sides of Eq. (7), we obtain

$$\ddot{x}+\;\Omega^2 \mathrm{x}=\Omega^2 \mathrm{x} - x^{-1/3}. \tag{8}$$

According to Eq. (4), the iteration scheme of Eq. (8) is

$$\ddot{x}_{k+1}+\Omega_k^2 x_{k+1}=\Omega_k^2 x_k-x_k^{-1/3}. \tag{9}$$

The first approximation $x_1(t)$ and the frequency Ω_0 will be obtained by putting $k=0$ in Eq. (9) and using Eq. (5), we obtain

$$\ddot{x}_1+\Omega_0^2 x_1=\Omega_0^2 A\cos\theta-(A\cos\theta)^{-1/3}. \tag{10}$$

where $\theta = \Omega_0 t$.

Now expanding $(\cos\theta)^{-1/3}$ in a Fourier Cosine series in the interval $[0, \pi]$ then Eq. (10) reduces to

$$\ddot{x}_1 + \Omega_0^2 x_1 = \Omega_0^2 A\cos\theta - \frac{1}{A^{1/3}}\big(1.426348\cos\theta - 0.713174\cos 3\theta + 0.509410\cos 5\theta - 0.407528\cos 7\theta + 0.344831\cos 9\theta - 0.301728\cos 11\theta\big). \tag{11}$$

No secular term in the solution for $x_1(t)$ requires that the coefficient of the $cos\theta$ term be zero from the right hand side of the Eq. (11). Thus we have

$$\Omega_0 = \frac{1.194298}{A^{2/3}}. \tag{12}$$

Then solving Eq. (11) and satisfying the initial condition $x_1(0) = A$, we obtain

$$x_1(t) = A\big(1.052312\cos\theta - 0.0625\cos 3\theta + 0.014880\cos 5\theta - 0.005952\cos 7\theta + 0.003021\cos 9\theta - 0.001762\cos 11\theta\big). \tag{13}$$

This is the first approximate solution of Eq. (8) and the related Ω_1 is to be determined.

The value of Ω_1 will be obtained from the solution of

$$\ddot{x}_2 + \Omega_1^2 x_2 = \Omega_1^2\, x_1 - x_1^{-1/3}. \tag{14}$$

Substituting $x_1(t)$ from Eq. (13) into the right hand side of Eq. (14), we obtain

$$\begin{aligned}\ddot{x}_2 + \Omega_1^2 x_2 = {} & \Omega_1^2 A\big(1.052312\cos\theta - 0.0625\cos 3\theta + 0.014880\cos 5\theta \\ & - 0.005952\cos 7\theta + 0.003021\cos 9\theta - 0.001762\cos 11\theta\big) \\ & - \frac{1}{A^{1/3}}\big(1.052312\cos\theta - 0.0625\cos 3\theta + 0.014880\cos 5\theta \\ & - 0.005952\cos 7\theta + 0.003021\cos 9\theta - 0.001762\cos 11\theta\big)^{-1/3}.\end{aligned} \tag{15}$$

$$\begin{aligned}\ddot{x}_2 + \Omega_1^2 x_2 = {} & \Omega_1^2 A\big(1.052312\cos\theta - 0.0625\cos 3\theta + 0.014880\cos 5\theta \\ & - 0.005952\cos 7\theta + 0.003021\cos 9\theta - 0.001762\cos 11\theta\big) \\ & - \frac{1}{A^{1/3}}\big(1.38767\cos\theta - 0.644734\cos 3\theta + 0.454911\cos 5\theta \\ & - 0.362239\cos 7\theta + 0.306038\cos 9\theta - 0.267895\cos 11\theta\big).\end{aligned} \tag{16}$$

The elimination of secular term from the Eq. (16), we obtain

$$\Omega_1 = \frac{1.14834}{A^{2/3}}. \tag{17}$$

Then solving Eq. (16) and satisfying initial condition. We obtain the second approximate solution,

$$x_2(t) = A\big(1.044512\cos\theta - 0.053302\cos 3\theta + 0.013203\cos 5\theta \\ -0.005598\cos 7\theta + 0.002863\cos 9\theta - 0.001678\cos 11\theta\big). \tag{18}$$

The third approximation x_3 and the value of Ω_2 will be obtained from the solution of

$$\ddot{x}_3 + \Omega_2^2 x_3 = \Omega_2^2\, x_2 - x_2^{-1/3}. \tag{19}$$

Substituting $x_2(t)$ from Eq. (18) into the right-hand side of Eq. (19), we obtain

$$\begin{aligned}\ddot{x}_3 + \Omega_2^2 x_3 = {}& \Omega_2^2 A\big(1.044512\cos\theta - 0.053302\cos 3\theta + 0.013203\cos 5\theta \\ & -0.005598\cos 7\theta + 0.002863\cos 9\theta - 0.001678\cos 11\theta\big) \\ & -\frac{1}{A^{1/3}}(1.044512\cos\theta - 0.053302\cos 3\theta + 0.013203\cos 5\theta \\ & -0.005598\cos 7\theta + 0.002863\cos 9\theta - 0.001678\cos 11\theta)^{-1/3}.\end{aligned} \tag{20}$$

$$\begin{aligned}\ddot{x}_3 + \Omega_2^2 x_3 = {}& \Omega_2^2 A\big(1.044512\cos\theta - 0.053302\cos 3\theta + 0.013203\cos 5\theta \\ & -0.005598\cos 7\theta + 0.002863\cos 9\theta - 0.001678\cos 11\theta\big) \\ & -\frac{1}{A^{1/3}}(1.39292\cos\theta - 0.65307\cos 3\theta + 0.460341\cos 5\theta \\ & -0.366474\cos 7\theta + 0.309581\cos 9\theta - 0.270977\cos 11\theta).\end{aligned} \tag{21}$$

Secular terms can be eliminated if the coefficient of the $cos\theta$ term is set to be zero from the Eq. (21), we obtain

$$\Omega_2 = \frac{1.154799}{A^{2/3}}. \tag{22}$$

Thus $\Omega_0, \Omega_1, \Omega_2, ...$ respectively obtained by Eq. (12), (17), (22),…. represents the approximation of frequencies of oscillator (7).

3 Results and Discussion

An iterative technique is presented to obtain approximate solution of inverse cubic nonlinear oscillator. In order to test the accuracy of the modified approach of iteration method, we compare our results with the other existing results from different methods. To show the accuracy, we have calculated the percentage errors (denoted by Er%) by the definition

$$\left|\frac{\Omega_e(A) - \Omega_i(A)}{\Omega_e(A)} \times 100\right|, \text{ where } i = 0, 1, 2 \cdots\cdots$$

We have used a modified iteration method to obtaining approximate solutions of the above oscillator. It has been shown that, in most of the cases our solution gives significant by better result than other existing results.

Herein we have calculated the first, second and third approximate frequencies which are denoted by Ω_0, Ω_1, and Ω_2 respectively. All the results are given in Table 1, to compare the approximate frequencies. We have also given the existing results determined by Mickens iteration method [26] and Mickens HB method [26].

Table 1. Comparison of the approximate frequencies with exact frequency Ω_e [26] of $\ddot{x} + x^{-1/3} = 0$.

Exact frequency Ω_e	$\frac{1.154700}{A^{2/3}}$		
Amplitude A	First approximate frequency Ω_0 Er(%)	Second approximate frequency Ω_1 Er(%)	Third approximate frequency Ω_2 Er(%)
Mickens iteration method [26]	$\frac{1.08148}{A^{2/3}}$ 6.3	$\frac{1.07634}{A^{2/3}}$ 6.78	$\frac{0.988591}{A^{2/3}}$ 14.38
Mickens HB method [26]	$\frac{1.31329}{A^{2/3}}$ 13.7	$\frac{1.18824}{A^{2/3}}$ 2.9	–
Adopted method	$\frac{1.19429}{A^{2/3}}$ 3.43	$\frac{1.14834}{A^{2/3}}$ 0.55	$\frac{1.154799}{A^{2/3}}$ 0.0085

In our study in the above table, it is seen that the third-order approximate frequency obtained by adopted method is almost same with exact frequency. It is found that, in each of the cases our solution gives significantly better result than other existing results. The compensation of this method consists of its simplicity, computational efficiency and convergence. It is also observed that the Mickens' iteration technique is convergent for this oscillator.

3.1 Convergence and consistency analysis

The basic idea of iteration methods is to construct a sequence of solutions x_k (as well as frequencies Ω_k) that have the property of convergence

$$x_e = \lim_{k \to \infty} x_k \quad \text{or,} \quad \Omega_e = \lim_{k \to \infty} \Omega_k$$

Here x_e is the exact solution of the given nonlinear oscillator.

In our technique, it has been shown that the solution gives the less error in each iterative step compared to the previous iterative step and finally $|\Omega_2 - \Omega_e| = |1.154799 - 1.154700| < \varepsilon$, where ε is a small positive number and A is chosen to be unity. From this, it is clear that the adopted method is convergent.

An iterative method of the form represented by Eq. (4) with initial guesses given in Eq. (5) and Eq. (6) is said to be consistent if $\lim_{k \to \infty} |x_k - x_e| = 0$ or, $\lim_{k \to \infty} |\Omega_k - \Omega_e| = 0$.

In the present study we observe that

$$\lim_{k \to \infty} |\Omega_k - \Omega_e| = 0 \text{ as } |\Omega_2 - \Omega_e| \approx 0.$$

Thus the consistency of the method is achieved.

4 Conclusion

In this work, we used a simple but effective modification of the iteration method to handle strongly nonlinear oscillators. An example is given to illustrate the effectiveness and convenience of this method. The results anticipated were compared with the others method. The obtained results show that the modification of the iteration method is more accurate than others method and this method is valid for large region.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Nayfeh AH. Perturbation method. Wiley, New York; 1973.

[2] Nayfeh AH. Introduction to perturbation techniques. Wiley, New York; 1981.

[3] Krylov NN, Bogoliubov NN. Introduction to nonlinear mechanics. Princeton University Press, New Jersey; 1947.

[4] Bogoliubov NN, Mitropolskii Yu. Asymptotic methods in the theory of nonlinear oscillations. Gordan and Breach, New York; 1961.

[5] Haque BMI, Alam MS, Rahman H. Perturbation solutions of fourth order damped oscillatory nonlinear systems. International Journal of Basic & Applied Sciences IJBAS/IJENS. 2011;11:3.

[6] Haque BMI, Rahman H, Ali Akbar M. Asymptotic solution of fourth order critically damped nonlinear systems under some special conditions. Journal of Engineering Science. 2010;1:95-104.

[7] Rahman H, Haque BMI, Ali Akbar M. Analytical solutions for fourth order damped-oscillatory nonlinear systems. Research Journal of Mathematics and Statistics. 2011;3(1):12-19.

[8] Rahman H, Haque BMI, Ali Akbar M. Asymptotic solutions of fourth order near critically damped nonlinear systems. Journal of Informatics and Mathematical Sciences. 2009;1:61-73.

[9] Nayfeh AH, Mook DT. Nonlinear oscillation. John Wiley & Sons, New York; 1979.

[10] Ramos JI. Approximate methods based on order reduction for the periodic solutions of nonlinear third-order ordinary differential equations. Appl. Math. Comput. 2010;215:4304-4319.

[11] Lakrad F, Belhaq M. Periodic solutions of strongly nonlinear oscillators by the multiple scales method. J. Sound Vib. 2002;258:677-700.

[12] Xu H, Cang J. Analysis of a time fractional wave-like equation with the homotopy analysis method. Phys. Lett. A. 2008;372:1250-1255.

[13] Turkyilmazoglu M. Analytic approximate solutions of rotating disk boundary layer flow subject to a uniform suction or injection. Int. J. of Mech. Sci. 2010;52(12):1735-1744.

[14] Turkyilmazoglu M. An optimal analytic approximate solution for the limit cycle of Duffing-van der Pol equation. J. Appl. Mech. 2011;78(2):021005.

[15] Turkyilmazoglu M. An effective approach for approximate analytical solutions of the damped Duffing equation. Physica Scripta. 2012;86(1):015301.

[16] Mickens RE. Comments on the method of harmonic balance. J. Sound Vib. 1984;94:456-460.

[17] Wu BS, Sun WP, Lim CW. An analytical approximate technique for a class of strongly nonlinear oscillator. Int. J. Nonlinear Mech. 2006;41:766-774.

[18] Gottlieb HPW. Harmonic balance approach to limit cycle for nonlinear jerk equation. J. Sound Vib. 2006;297:243-250.

[19] Hosen MA. Accurate approximate analytical solutions to an anti-symmetric quadratic nonlinear oscillator. African Journal of Mathematics and Computer Science Research. 2013;6(5):77-81.

[20] Belendez A, Pascual C, Belendez T, Hernandez A. Solution of an anti-symmetric quadratic nonlinear oscillator by a modified He's homotopy perturbation method. Nonlinear Anal. Real World Appl. 2009;10(1):416-427.

[21] Borges CA, Cesari L, Sanchez DA. Quarterly of Applied Mathematics. 1975;32:457.

[22] Miletta P, Chuagui R. Analysis, Geometry and probability. Marcel Dekker, New York; 1985.

[23] Bobylev NA, Burman YM, Korovin SK. Approximation procedures in nonlinear oscillation theory. Walter deGruyter, Berlin; 1994.

[24] Mickens RE. Iteration procedure for determining approximate solutions to nonlinear oscillator equation. J. Sound and Vib. 1987;116:185-188.

[25] Mickens RE. Nonlinear oscillations. Cambridge University Press, New York; 1991.

[26] Mickens RE. Truly nonlinear oscillations. World Scientific, Singapore; 2010.

[27] Hu H, Tang JH. A classical iteration procedure valid for certain strongly nonlinear oscillator. J. Sound Vib. 2006;299:397-402.

[28] Haque BMI, Alam MS, Majedur Rahman M. Modified solutions of some oscillators by iteration procedure. J. Egyptian Math. Soci. 2013;21:68-73.

[29] Haque BMI, Alam MS, Majedur Rahman M, Yeasmin IA. Iterative technique of periodic solutions to a class of non-linear conservative systems. Int. J. Conceptions on Computation and Information Technology. 2014;2(1):92-97.

[30] Haque BMI. A new approach of modified Mickens iteration method for solving some nonlinear jerk equations. British Journal of Mathematics & Computer Science. 2014;4:22.

[31] Lim CW, Wu BS. A modified procedure for certain nonlinear oscillators. J. Sound Vib. 2002;257: 202-206.

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