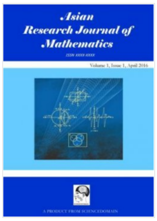

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Limit and Continuity Revisited via Convergence

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Commentary

Abstract

We derive some of the standard results on limits of elementary functions defined on subsets of real-line, whose rigorous proofs are often avoided in the routine teaching and learning of calculus. For proofs, we essentially follow the Weierstrass's systematized modern formalization of Cauchy's idea of transforming the concept of limit into "the algebra of inequalities".

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1 Introduction

Calculus has found highest place in modern science since its main discoveries by Newton and Leibniz, and its rigorous development by Cauchy, Weierstrass, and Riemann [1, 2]. In fact, the essence of calculus is well described in the following mesmerizing quote by John von Neumann [3].

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> "The calculus was the first achievement of modern mathematics and it is difficult to overestimate its importance. I think it defines more unequivocally than anything else the inception of modern mathematics, and the system of mathematical analysis, which is its logical development, still constitutes the greatest technical advance in exact thinking."

However, the students often find the subject difficult. One reason behind their difficulty may be that the rigorous treatment of the subject matter is often avoided by their calculus teacher during school and college studies. For instance, the students often come across the following simple calculation on limit, when the calculus is just introduced to them.

$$\lim_{\substack{x \to 1 \\ (x \neq 1)}} \frac{x^2 - 1}{x - 1} = \lim_{\substack{x \to 1 \\ (x \neq 1)}} \frac{(x-1)(x+1)}{x-1} = \lim_{\substack{x \to 1 \\ (x \neq 1)}} x + 1 = (x = 1) + 1 = 2, \tag{1.1}$$

where $x \neq 1$ is assumed until the last step, and then, suddenly, $x = 1$ is used to get the limit equal to 2. The use of $x = 1$ in (1.1) can be explained rigorously by using the $\epsilon - \delta$ definition of limit, that is, we may consider for a given real number $\epsilon > 0$, $|x + 1 - 2| = |x - 1| < \epsilon$ for $|x - 1| < \delta$, where we choose $\delta = \epsilon$. Alternatively, the function defined by $g(x) = x + 1$ is continuous at point $x = 1$ so that $\lim_{x \to 1} g(x) = g(1)$. However, here, the continuity of the function g must be established by a mean independent of the concept of limit, which can be done using Cauchy's idea of continuity via convergence.

As an another example, if a student is not exposed to the rigorous treatment of the subject, he never seems to have understood why $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$ is different from 1? or why it exists at all? The real problem arises when such a freshman enters University for graduate studies, and the proofs of many such standard limits are often presumed by his calculus Professor. So, even after completing his post-graduation in Mathematics, he lacks the firm foundation in calculus. In this regard, the present exposition may be helpful to the one who is looking for the proofs of standard results on limits at one place, which otherwise are not done in lectures, and many of them are left as exercises in the textbooks on calculus.

Throughout, the reader is assumed to have some familiarity with the algebraic properties and the order properties of real-line such as "the least upper bound property" and "the archimedean property". The symbols $\mathbb{N}$, $\mathbb{Q}_+$, $\mathbb{Q}$, $\mathbb{R}_+$, and $\mathbb{R}$ denote the set of positive integers, the set of all positive rational numbers, the set of all rational numbers, the set of all positive real numbers, and the set of all real numbers, respectively. A subset A of $\mathbb{R}$ is said to be bounded above if there is a real number M, such that $a \leq M$ for all $a \in A$. Such a real number M is called an upper bound of the set A, and a real number α is said to be supremum of the set A (provided it exists) denoted $\sup A$ if (i) α is an upper bound of A, and (ii) if $\beta < \alpha$ then, β is not an upper bound of A. The set A is said to be bounded below if there is a real number m, such that $m \leq a$ for all $a \in A$. Such a real number m is called a lower bound of the set A, and a real number γ is said to be infimum of A (provided it exists) denoted $\inf A$ if (i) γ is a lower bound of A, and (ii) if $\gamma < \mu$ then, μ is not a lower bound of A. A subset of $\mathbb{R}$ is called bounded if it is both bounded above as well as bounded below. By the completeness property of real-line, if a set of real numbers A is bounded above (bounded below), then, $\sup A$ ($\inf A$) always exists in $\mathbb{R}$. The archimedean property says that for any two positive real numbers x and y satisfying $x < y$, there exists a positive integer n, such that $y < nx$ [4, 5].

If $S \subset \mathbb{R}$, which is bounded above, then the real number $\sup S$ is unique. To see this, let $\alpha = \sup S$ and $\beta = \sup S$. If $\alpha < \beta$, then, β being $\sup S$, α is not an upper bound of S, which is absurd since $\alpha = \sup S$. So, $\alpha \geq \beta$. On applying the same argument after interchanging α and β, we will get $\beta \geq \alpha$. So, $\alpha = \beta$. Similarly, if $\inf S$ exists then it is unique.

Before proving the next result, we will establish equality of given two real numbers via an inequality. We assert that for any $x, y \in \mathbb{R}$, $x = y$ if and only if the difference $|x-y|$ can be made as small as we wish, that is, for every $\epsilon > 0$, $|x-y| < \epsilon$. For proof, if $x = y$ then $|x-y| = 0 < \epsilon$ holds. Conversely, suppose $|x-y| < \epsilon$ for every $\epsilon > 0$. If possible, let $|x-y| > 0$. We take $\epsilon = |x-y|$ and apply the given hypothesis to get $|x-y| < \epsilon = |x-y|$ or $1 < 1$, which is absurd. So, $x = y$.

We also need the Bernoulli inequality, which we state as follows. For any real number $t > -1$,

$$(1+t)^n \geq (1+nt), \tag{1.2}$$

for all $n \in \mathbb{N}$, which can be proved using induction on n.

2 Preliminaries

Theorem 2.1 **(Existence of** n**th root).** *Let $x \in \mathbb{R}_+$, and let $n \in \mathbb{N}$. Then there exists exactly one $y \in \mathbb{R}_+$, such that $y^n = x$.*

Proof. Let $S = \{y \mid y^n \leq x,\ y \in \mathbb{R}_+\}$. As $0 < \frac{x}{1+x} < 1$, we have $\left(\frac{x}{1+x}\right)^n < \frac{x}{1+x} < x$, which shows that $\left(\frac{x}{1+x}\right) \in S$. So, $S \neq \emptyset$. By Bernoulli's inequality, $(1+x)^n \geq (1+nx) \geq (1+nt^n) > nt^n > t^n$ for all $t \in S$, which shows that the set S is bounded above. By the least upper bound property, let $\alpha = \sup S$. Note that $\alpha > 0$. We will show that $\alpha^n = x$. So, let $\epsilon > 0$ be given. It will be enough to show that $|x - \alpha^n| < \epsilon$.

If k is any positive integer satisfying $\alpha > \frac{1}{k}$, we will show that $|\frac{x}{\alpha^n} - 1| < \frac{c}{k}$, for some fixed positive real number c. For proof, observe that $\left(\alpha - \frac{1}{k}\right)$ is not an upper bound of S. So, there is some $t \in S$, such that $0 < \left(\alpha - \frac{1}{k}\right) < t$, which gives $\left(\alpha - \frac{1}{k}\right)^n < t^n \leq x$, that is, $\left(\alpha - \frac{1}{k}\right) \in S$. Also, $\alpha < \left(\alpha + \frac{1}{k}\right)$, which implies $\left(\alpha + \frac{1}{k}\right) \notin S$. So, $\left(\alpha + \frac{1}{k}\right)^n > x$. Now we have $\left(\alpha - \frac{1}{k}\right)^n < x < \left(\alpha + \frac{1}{k}\right)^n$ or $\left(1 - \frac{1}{k\alpha}\right)^n < \frac{x}{\alpha^n} < \left(1 + \frac{1}{k\alpha}\right)^n$. Thus, $-\frac{g(k)}{k} < -\frac{g(-k)}{k} < \left(\frac{x}{\alpha^n} - 1\right) < \frac{g(k)}{k}$, where we define $g(k) = k\{\left(1 + \frac{1}{k\alpha}\right)^n - 1\} > 0$, which satisfies $g(-k) < g(k)$, and $g(k) = \frac{1}{\alpha}\sum_{i=1}^{n}\binom{n}{i}\frac{1}{(k\alpha)^{i-1}} < \frac{1}{\alpha}\sum_{i=1}^{n}\binom{n}{i}1 = g(1/\alpha)$, since $\frac{1}{k\alpha} < 1$. So, we have $\left|\frac{x}{\alpha^n} - 1\right| < \frac{c}{k}$, where we take $c = g(1/\alpha)$.

By archimedean property, choose a positive integer N, such that $\frac{g(1/\alpha)}{N} < \epsilon$ and $\alpha > \frac{1}{N}$. Then we have $\left|\frac{x}{\alpha^n} - 1\right| < \frac{g(1/\alpha)}{N} < \epsilon$. So, $\frac{x}{\alpha^n} = 1$ or $\alpha^n = x$.

Now, if β is another positive real number, such that $\beta^n = x$, then it is clear that $\beta = \sup S \Rightarrow \alpha = \beta$. □

Definition 2.1 **(nth positive root).** For $x \in \mathbb{R}_+$ and $n \in \mathbb{N}$, if $\alpha \in \mathbb{R}_+$, such that $\alpha^n = x$, we call $\alpha = \sqrt[n]{x}$ the nth positive root of x. Further, if $\frac{p}{q} \in \mathbb{Q}, q > 0$, we define the rational exponentiation of x by the rational number $\frac{p}{q}$ to be the real number $x^{\frac{p}{q}} = (\sqrt[q]{x})^p$.

Let x be a positive real number, and let $n \in \mathbb{N}$. Let y be the positive real number, such that $y = \sqrt[n]{x}$. By the above definition, $y^n = x$, and we have

$$(x-1) = (y^n - 1) = (y-1)(y^{n-1} + y^{n-2} + \ldots + y + 1), \tag{2.1}$$

which shows that $x > 1$ if and only if $y > 1$, and $0 < x < 1$ if and only if $0 < y < 1$, since $(y^{n-1} + y^{n-2} + \ldots + y + 1) > 0$ for $y > 0$.

Also note that $x^{n/n} = (\sqrt[n]{x})^n = \alpha^n = x$. If $a = \frac{p}{q}$ and $b = \frac{m}{n}$, where $|p|$, $|m|$, q, $n \in \mathbb{N}$, then $x^{a+b} = x^{\frac{pn+qm}{qn}} = (\sqrt[qn]{x})^{pn+qm} = (\sqrt[qn]{x})^{pn}(\sqrt[qn]{x})^{qm} = x^{\frac{pn}{qn}}x^{\frac{qm}{qn}} = x^{\frac{p}{q}}x^{\frac{m}{n}} = x^a x^b$. Also, if $x^{\frac{p}{q}} = y$ then $y^q = ((\sqrt[q]{x})^p)^q = ((\sqrt[q]{x})^{pq} = ((\sqrt[q]{x})^q)^p = x^p$. Now if we let $x^a = y$ and $y^b = z$ so that $z = (x^a)^b$

then, $x^p = y^q$ and $y^m = z^n$, which implies $x^{pm} = y^{qm} = z^{qn}$, from which we obtain $z = x^{\frac{pm}{qn}} = x^{ab}$. Thus, the laws of rational exponentiation $x^{a+b} = x^a x^b$ and $(x^a)^b = x^{ab}$ hold for all $a, b \in \mathbb{Q}$. It is also easy to show that for $x, y \in \mathbb{R}_+$, $(xy)^a = x^a y^a$ for all $a \in \mathbb{Q}$.

2.1 Convergence

A sequence of real numbers is a map $x : \mathbb{N} \to \mathbb{R}$, where for each n, $x(n)$ denoted x_n is called the n-th term of the sequence. The sequence x is denoted by $\{x_n\}$. A sequence $\{x_n\}$ is said to be (i) monotonically increasing if $x_n \leq x_{n+1}$ and (ii) monotonically decreasing if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$. A sequence which is either monotonically increasing or monotonically decreasing, is called a monotonic sequence. A strictly increasing (decreasing) sequence can be defined in a similar way by requiring strict inequality $<(>)$ between its consecutive terms.

Definition 2.2 **(Convergence).** A sequence of real numbers $\{x_n\}$ is said to converge to a real number x, if for a given $\epsilon > 0$, there is a positive integer N, such that $|x_n - x| < \epsilon$ for all $n \geq N$.

If the sequence of real numbers $\{x_n\}$ converges to x, we express this as $x_n \to x$ and write $x = \lim_{n\to\infty} x_n$.

Convergence of sequences in $\mathbb{R}$ is unique in the sense that if $x_n \to x$ and $x_n \to y$ in $\mathbb{R}$, then $x = y$. For proof, let $\epsilon > 0$ be given. Choose $N_1, N_2 \in \mathbb{N}$, such that $|x_n - x| < \epsilon/2$ for all $n \geq N_1$ and $|x_n - y| < \epsilon/2$ for all $n \geq N_2$. Define $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, we have $|x - y| = |x - x_n + x_n - y| \leq |x - x_n| + |y_n - y| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. So, $x = y$.

It is useful to note that in order to establish that the given sequence $\{x_n\}$ converges to the real number x, it is sufficient to take $0 < \epsilon < 1$ and find N corresponding to ϵ, such that$|x_n - x| < \epsilon$ for all $n \geq N$.

The following standard result can be proved easily.

Lemma 2.2. *Let $\{x_n\}$ and $\{y_n\}$ be two sequences in $\mathbb{R}$, and let $x, y \in \mathbb{R}$.*
(a) If $x_n \to x$ and $y_n \to y$ then $(x_n + y_n) \to x + y$, and $(cx_n) \to cx$ for any fixed $c \in \mathbb{R}$.
(b) If $x_n \to x$ and $y_n \to y$ then $(x_n y_n) \to xy$.
(c) If $x_n \to x$, and $x \neq 0$ then $\left(\frac{1}{x_n}\right) \to \frac{1}{x}$, where it is understood that $\frac{1}{x_n}$ is defined whenever $x_n \neq 0$.

Theorem 2.3. *A monotonic sequence of real numbers is convergent if and only if it is bounded.*

Proof. Without loss of generality, let $\{x_n\}$ be a monotonically increasing sequence of real numbers. If $x_n \to x$ for some $x \in \mathbb{R}$ then there is a $k \in \mathbb{N}$, such that $|x_n - x| < 1$ or $x - 1 < x_n < x + 1$ for all $n \geq k$. Define $m = \inf\{x - 1, x_1, \ldots, x_{k-1}\}$ and $M = \sup\{x - 1, x_1, \ldots, x_{k-1}\}$. Then $m \leq x_n \leq M$ for all n, which proves that $\{x_n\}$ is bounded.

Conversely, let $\{x_n\}$ be bounded. Let $\alpha = \sup\{x_n\}$, and let $\epsilon > 0$ be given. Then $\alpha - \epsilon < \alpha$, so that $\alpha - \epsilon$ is not an upper bound of $\{x_n\}$. Then there is a positive integer N, such that $\alpha - \epsilon < x_N \leq \alpha$. Since $x_N \leq x_{N+1} \leq \ldots$, and $x_n \leq \alpha$ for all n, we have $|x_n - \alpha| < \epsilon$ for all $n \geq N$, which proves that $x_n \to \alpha$. □

Remark 2.1. The proof of Theorem 2.3 shows that if $\{x_n\}$ is a monotonically increasing, bounded sequence of real numbers, then $x_n \to \sup_n\{x_n\}$. A similar observation can be made for a monotonically decreasing, bounded sequence $\{y_n\}$ of real numbers, which will converge to $\inf_n\{y_n\}$.

Example 2.4. *For $0 < x < 1$ and $k \in \mathbb{N}$, consider the sequence $\{x_n\}$, $n \in \mathbb{N}$, where $x_n = n^k x^n$. Note that $n^k x^n - (n+1)^k x^{n+1} = n^k x^n \left(1 - \left(1 + \frac{1}{n}\right)^k x\right) > 0$ for $n > N_x := \frac{\sqrt[k]{x}}{1-\sqrt[k]{x}}$, which proves that the given sequence is strictly decreasing for all $n > N_x$, and is bounded below by 0. So, by Theorem 2.3, the given sequence is convergent.*

Alternatively, let $x = \frac{1}{(1+y)^{k+1}}$, and consider the following using (1.2)

$$(1+y)^{n(k+1)} \geq (1+ny)^{k+1} > (ny)^{k+1}, \tag{2.2}$$

which gives $0 < n^k(1+y)^{-n(k+1)} < \frac{1}{y^{k+1}}\frac{1}{n}$ or $0 < n^k x^n < M(k,x)\frac{1}{n}$, where

$$M(k,x) = \frac{x}{(1 - x^{1/(k+1)})^{k+1}} > 0.$$

Now for any $\epsilon > 0$, choose $N \in \mathbb{N}$, such that $M(k,x) < N\epsilon$. Then for all $n \geq N$, $n^k x^n < M(k,x)\frac{1}{N} < \epsilon$. So, $x_n \to 0$.

Example 2.5. *The sequence $\{x^n\}$, where $0 < |x| < 1$ converges to 0. For proof, let $\epsilon > 0$ be given. Since $(1-|x|) > 0$, by archimedean property, find $N \in \mathbb{N}$, such that $0 < \frac{1}{n} < (1-|x|)\epsilon$ for all $n \geq N$. Now use Bernoulli's inequality to obtain $\frac{1}{|x|^n} = \left(1 + \frac{1}{|x|} - 1\right)^n \geq 1 + n\left(\frac{1}{|x|} - 1\right) > n\left(\frac{1}{|x|} - 1\right)$. So, we have $0 < |x^n| < \frac{|x|}{n(1-|x|)} < \frac{1}{n(1-|x|)} < \epsilon$ for all $n \geq N$. Thus, $x^n \to 0$.*

Example 2.6. *Let $r \in \mathbb{R}$, where $0 < |r| < 1$, and let $x_n = 1 + r + \ldots + r^{n-1}$ for each $n \in \mathbb{N}$. Since $r^n \to 0$, choose $N \in \mathbb{N}$, such that $|r^n| < (1-r)\epsilon$ for all $n \geq N$. Now consider the following for $n \geq N$, $|x_n - \frac{1}{1-r}| = |\sum_{m=0}^{n-1} r^m - \frac{1}{1-r}| = |\frac{1-r^n}{1-r} - \frac{1}{1-r}| = \frac{|r|^n}{1-r} < \frac{(1-r)\epsilon}{1-r} = \epsilon$. So, $x_n \to \frac{1}{1-r}$.*

2.2 Continuity and limit

Let A and B be two nonempty subsets of $\mathbb{R}$.

Definition 2.3 (Continuity). A map $f : A \to B$ is said to be continuous at a point $x \in A$ if for every sequence $\{x_n\}$ in A, such that $x_n \to x$ implies $f(x_n) \to f(x)$.

Definition 2.4 (Limit point). A point $p \in \mathbb{R}$ is said to be a limit point of the set A if for every $\delta > 0$, there is at least one $x \in A$, such that $0 < |x - p| < \delta$.

Definition 2.5 (Limit). Let $a \in \mathbb{R}$ be a limit point of A. The map $f : A \to B$ is said to have a limit L at the point a if for every $\epsilon > 0$, there exists a real number $\delta > 0$, such that $x \in A$ and $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$. We express this as $L = \lim_{x\to a} f(x)$.

The map $f : A \to B$ is said to have L as its left hand limit at the point $a \in \mathbb{R}$ if for every $\epsilon > 0$, there exists a real number $\delta > 0$, such that $x \in A$ and $-\delta < (x - a) < 0$ implies $|f(x) - L| < \epsilon$. We express this as $L = \lim_{x\to a^-} f(x)$. Similarly, f is said to have L as its right hand limit at a if for every $\epsilon > 0$, there exists a real number $\delta > 0$, such that $x \in A$ and $0 < (x - a) < \delta$ implies $|f(x) - L| < \epsilon$, which is expressed as $L = \lim_{x\to a^+} f(x)$.

Observe from the above definitions that $\lim_{x\to a} f(x)$ exists if and only if both $\lim_{x\to a^-} f(x)$ as well as $\lim_{x\to a^+} f(x)$ exist and are equal.

Also note that limit of a function is defined only at a limit point of its domain set. To this, the following equivalence of continuity and limit can be obtained.

Theorem 2.7. (Continuity via Limit) *Let $a \in \mathbb{R}$ be a limit point of A and $a \in A$. The function $f : A \to B$ is continuous at the point a if and only if $\lim_{x\to a} f(x) = f(a)$.*

Proof. Let f be continuous. If possible, let there be some $\epsilon > 0$, such that for any $\delta > 0$, there is some point $x_\delta \in A$, which satisfies $0 < |x_\delta - a| < \delta$ but $|f(x_\delta) - f(a)| \geq \epsilon$. In particular, for this $\epsilon > 0$ and for any $n \in \mathbb{N}$, there exist $x_n \in A$, such that $|x_n - a| < \frac{1}{n}$ but $|f(x_n) - f(a)| \geq \epsilon$, which shows that $x_n \to a$ but $f(x_n) \not\to f(a)$ thereby contradicting the continuity of f at a.

Conversely, let for a given $\epsilon > 0$, there is a real number $\delta > 0$, such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$. Let $\{x_n\}$ be a sequence of points of A converging to a. Choose $N \in \mathbb{N}$, such that $|x_n - a| < \delta$ for all $n \geq N$. By the hypothesis, we have $|f(x_n) - f(a)| < \epsilon$ for all $n \geq N$, that is, $f(x_n) \to f(a)$ as desired. □

Definition 2.6. Let S be a nonempty subset of $\mathbb{R}$, and let $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$. Define $(f + g) : S \to \mathbb{R}$ and $(fg) : S \to \mathbb{R}$ to be the sum and product of the functions f and g, respectively, such that $(f + g)(x) = f(x) + g(x)$ and $(fg)(x) = f(x)g(x)$ for all $x \in S$. We also define for any $c \in \mathbb{R}$, the function $(cf) : S \to \mathbb{R}$, where $(cf)(x) = cf(x)$. If $g(x) \neq 0$ for all $x \in S$, we define $\frac{1}{g} : S \to \mathbb{R}$ by $\frac{1}{g}(x) = \frac{1}{g(x)}$.

It is easy to observe that if $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$, then $\lim_{x\to a}(f+g)(x) = L_1+L_2$ and $\lim_{x\to a}(fg)(x) = L_1L_2$. Also, if $L_2 \neq 0$, then $\lim_{x\to a}\left(\frac{1}{g}\right)(x) = \frac{1}{L_2}$. From this discussion, the following results can be easily deduced.

Theorem 2.8. *Let S be a nonempty subset of $\mathbb{R}$, and let $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ be two continuous functions. Then*

(a) *the functions $f + g$ and cf for any fixed $c \in \mathbb{R}$ are both continuous.*
(b) *the function fg is continuous.*
(c) *if $g(x) \neq 0$ for all $x \in S$ then the function $\frac{1}{g}$ is continuous.*

Let $f : A \to B$, and $g : B \to C$ be two continuous maps, where $A, B, C \subseteq \mathbb{R}$. Then the composite map $(g \circ f) : A \to C$, where $(g \circ f)(x) = g(f(x))$ is continuous. For proof, let $x \in S$, and let $x_n \to x$ in S. By continuity of f, we have $f(x_n) \to f(x)$. Now by continuity of g, we have $(g \circ f)(x_n) = g(f(x_n)) \to g(f(x)) = (g \circ f)(x)$. So, $g \circ f$ is continuous. Now we have the following.

Theorem 2.9. *Let $A, B, C \subset \mathbb{R}$, and let $f : A \to B$. If $g : B \to C$ is continuous, and $\lim_{x\to a} f(x) = L \in B$ for some $a \in \mathbb{R}$, then $\lim_{x\to a}(g \circ f)(x)$ exists. Moreover, we have $\lim_{x\to a}(g \circ f)(x) = g(\lim_{x\to a} f(x))$.*

3 Revisiting Continuity and Limit Via Convergence

Lemma 3.1. *If $p_n = \frac{1}{2^n}$ for all $n \in \mathbb{N}$, then for any real number x, $\sin(p_n x) \to 0$.*

Proof. First note from Bernoulli inequality that $2^n = (1 + 1)^n \geq (1 + n)$ or $2^n > n$. So, $p_n < \frac{1}{n}$ for all $n \in \mathbb{N}$. Also, for every $\epsilon > 0$, by archimedean property of $\mathbb{R}$, there is a positive integer N, such that $\frac{1}{N} < \epsilon$. Since $p_N < \frac{1}{N}$, we have $p_N < \epsilon$. We have shown that for any $\epsilon > 0$, there exists a positive integer N, such that $p_N < \epsilon$.

Now for $-\pi/2 \leq x < y \leq \pi/2$, we have $(\sin x - \sin y) = -2\sin\left(\frac{y-x}{2}\right)\cos\left(\frac{y+x}{2}\right) < 0$, since $(y - x) \in [0, \pi]$ and $(y + x) \in [-\pi, \pi]$, which proves that sin function is strictly increasing on $[-\pi/2, \pi/2]$. Choose $N \in \mathbb{N}$, such that $p_N|x| < 1 < \pi/2$. So, $1 > \sin(p_N|x|) > \sin(p_{N+1}|x|) > \ldots > 0$. By Remark after Theorem 2.3, the sequence $\{\sin(p_{n+N}|x|)\}$ converges to $\beta = \inf_{n\in\mathbb{N}}\{\sin(p_{n+N}|x|)\}$. Clearly, $\beta \geq 0$. If possible, let $0 < \beta < 1$. Since $\sin : (0, \pi/2) \to (0, 1)$ is invertible, choose $\alpha \in (0, \pi/2)$, such that $\sin\alpha = \beta$. Choose m large enough, such that $p_m|x| < \alpha$. So, $\sin(p_{m+N}|x|) < \sin\alpha = \beta$, which is a contradiction to the fact that β is the inf of the set $\{\sin(p_{n+N}|x|) \mid n \in \mathbb{N}\}$. Thus, $\beta = 0$, which proves that $\sin(p_n x) \to 0$. □

Theorem 3.2. *(a)* $\lim_{x\to 0} \sin x = 0$.
(b) If $\{x_n\}$ *is a sequence of real numbers, such that* $x_n \to x$ *then* $\sin(x_n) \to \sin x$.
(c) sin *function is continuous.*
(d) cos *function is continuous.*

Proof. Let $\epsilon > 0$ be given. By Lemma 3.1, choose $N \in \mathbb{N}$, such that $\sin p_n < \epsilon$ for all $n \geq N$.

Now, $0 < |t| < p_N < 1 < \frac{\pi}{2}$ implies $\sin|t| < \sin p_N < \epsilon$. But $\sin|t| = |\sin t|$ for all $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. So, $\lim_{t\to 0} \sin t = 0$, which proves (a).

For the proof of (b), observe that $|x_n - x| \to 0$. So, there is a positive integer k, such that $0 < |x_n - x| < p_N$, where N is as chosen in (a) for all $n \geq k$. By (a), we have

$$0 < |x_n - x| < p_N < 1 \;\Rightarrow\; \sin\Big(\frac{|x_n - x|}{2}\Big) < \sin\Big(\frac{p_N}{2}\Big) = \sin p_{N+1} < \epsilon, \tag{3.1}$$

for all $n \geq k$, as $\frac{p_N}{2} = \frac{1}{2^{N+1}} = p_{N+1}$. Now consider for $n \geq k$,

$$\begin{aligned} |\sin x_n - \sin x| &= \left|2\sin\left(\frac{x_n - x}{2}\right)\cos\left(\frac{x_n + x}{2}\right)\right| \\ &\leq 2\sin\left(\frac{|x_n - x|}{2}\right) \leq 2\sin\left(\frac{p_N}{2}\right) < 2\epsilon, \end{aligned} \tag{3.2}$$

where the last inequality follows from (3.1). So, $\sin x_n \to \sin x$, which proves (b).

By definition of continuity, (b) implies (c).

Finally, (d) follows from (c), since $\cos = \sin \circ f$, where $f(x) = \frac{\pi}{2} - x$, and composition of two continuous functions is continuous. □

Theorem 3.3. *(a) The function* $f : \mathbb{R} \to \mathbb{R}$, *where* $f(x) = x^k,\ k \in \mathbb{N}$ *is continuous.*
(b) The function $g : \mathbb{R}_+ \to \mathbb{R}$, *where* $g(x) = x^{1/k},\ k \in \mathbb{N}$ *is continuous.*

Proof. (a) Let $x_n \to x$ in $\mathbb{R}$. By Lemma 2.2(b), $x_n^k \to x^k$ or $f(x_n) \to f(x)$. So, the map f is continuous at x.

(b) Let $\epsilon > 0$ be given, and let $x \in \mathbb{R}_+$. Let $\{x_n\}$ be a sequence in $\mathbb{R}_+$, such that $x_n \to x$. So, we can choose $N \in \mathbb{N}$, such that $|x_n - x| < \epsilon x^{1-\frac{1}{k}}$ for all $n \geq N$. Also, $x - x_n = (x^{1/k} - x_n^{1/k})y_n$, where $y_n = \sum_{i=1}^{k} x^{\frac{k-i}{k}} x_n^{\frac{i-1}{k}}$. Since $x, x_n \in \mathbb{R}_+$, we have $y_n > x^{1-\frac{1}{k}}$ or $y_n^{-1} < x^{-1+\frac{1}{k}}$ for all n. With these, we have $|x^{1/k} - x_n^{1/k}| = |x - x_n| y_n^{-1} < |x - x_n| x^{-1+\frac{1}{k}} < \epsilon$, for all $n \geq N$. So, f is continuous at x. □

Corollary 3.4. *The function* $f : \mathbb{R}_+ \to \mathbb{R}$, *such that* $f(x) = x^r,\ r \in \mathbb{Q}$ *is continuous.*

Proof. Let $r = \frac{p}{q}$, where $p, q \in \mathbb{N}$. By Theorem 3.3(b), the map defined by $x \mapsto x^{1/q}$ is continuous, and by Theorem 3.3(a), the map defined by $y \mapsto y^p$ is continuous. Composing these maps gives the desired continuous function. □

Theorem 3.5. *Let* $r > 1$ *and* $0 < s < 1$ *be two rational numbers. Then for* $x > y > 0$, $x, y \in \mathbb{R}$, $rx^{r-1}(x - y) > x^r - y^r > ry^{r-1}(x - y)$ *and* $sx^{s-1}(x - y) < x^s - y^s < sy^{s-1}(x - y)$.

Proof. We will prove the result for the case $r > 1$. The inequality for $0 < s < 1$ can be proved the same way.

For $\alpha > 0$ and $q \in \mathbb{Q}_+$, let us define

$$f_q(\alpha) = \frac{\alpha^q - 1}{(\alpha - 1)q}. \tag{3.3}$$

First, we show that for $q \in \mathbb{N}$, $f_q(\alpha) < f_{q+1}(\alpha)$ for $\alpha > 1$, and $f_q(\alpha) > f_{q+1}(\alpha)$ for $0 < \alpha < 1$. For proof, we observe that for $\alpha > 1$, $f_q(\alpha) < \alpha^q$, and for $0 < \alpha < 1$, $f_q(\alpha) > \alpha^q$. These inequalities along with $f_{q+1}(\alpha) = \frac{1+\alpha+\ldots+\alpha^q}{q+1} = \frac{q}{q+1}\Big(f_q(\alpha) + \frac{\alpha^q}{q}\Big)$ establish the claim.

Now we show that for $u, v \in \mathbb{Q}_+$, $u < v$ implies $f_u(\alpha) < f_v(\alpha)$ for $\alpha > 1$, and $u < v$ implies $f_u(\alpha) > f_v(\alpha)$ for $0 < \alpha < 1$. So, let $u = \frac{p}{q}$ and $v = \frac{\ell}{m}$, where $p, q, \ell, m \in \mathbb{N}$. First assume that $\alpha > 1$. Now $u < v$ implies $\frac{p}{q} < \frac{\ell}{m}$, which gives $pm < q\ell$. Since $\alpha > 1$, $\sqrt[qm]{\alpha} > 1$. As proved earlier, we have $f_{pm}(\sqrt[qm]{\alpha}) < f_{q\ell}(\sqrt[qm]{\alpha})$, which gives $\frac{\alpha^{pm/(qm)}-1}{(\sqrt[qm]{\alpha}-1)pm} < \frac{\alpha^{q\ell/(qm)}-1}{(\sqrt[qm]{\alpha}-1)q\ell}$ or $\frac{\alpha^{p/q}-1}{p/q} < \frac{\alpha^{\ell/m}-1}{\ell/m}$. So, $f_u(\alpha) < f_v(\alpha)$, which proves the assertion for $\alpha > 1$. The proof for the case $0 < \alpha < 1$ follows the same way.

Next we establish that for $q \in \mathbb{Q}_+$, $q > 1$, we have $(\alpha^q - 1) < q\alpha^{q-1}(\alpha - 1)$ for all $\alpha > 0$. For proof, assume $\alpha > 1$. As proved in the preceding paragraph, we have $f_1(1/\alpha) > f_q(1/\alpha)$, which gives $(1-\alpha)q > \alpha^{1-q}(1-\alpha^q)$ or $(\alpha^q - 1) < q\alpha^{q-1}(\alpha-1)$. On the other hand if $0 < \alpha < 1$, again as before, $f_1(1/\alpha) < f_q(1/\alpha)$, which gives $(1-\alpha^q) > q\alpha^{q-1}(1-\alpha)$ as desired.

Finally, use the inequality $\alpha^q - 1 < q\alpha^{q-1}(\alpha - 1)$ for $\alpha = x/y$, $\beta = y/x$, and $q = r$ to obtain $x^r - y^r < rx^{r-1}(x-y)$ and $x^r - y^r > ry^{r-1}(x-y)$, respectively. □

Theorem 3.6. *The following results hold for any real number $a > 0$.*

(a) $a^{1/n} \to 1$.
(b) The function $f : \mathbb{Q} \to \mathbb{R}$, where $f(q) = a^q$ is continuous.
(c) For $x \in \mathbb{R}$, if $\{q_n\}$ is a sequence of rational numbers, such that $q_n \to x$, then the sequence $\{a^{q_n}\}$ converges.

Proof. (a) First assume that $a > 1$, so that $a^{1/n} > 1$ for all n, and the sequence $\{a^{1/n}\}$ is strictly decreasing. By Theorem 2.3, the sequence $\{a^{1/n}\}$ converges. Let $x_n = (a^{1/n} - 1)$. Using Bernoulli inequality, we have $a = (1+x_n)^n > (1+nx_n)$, since $x_n > 0$. So, $0 < x_n < \frac{a-1}{n}$ for all n. Now for a given $\epsilon > 0$, choose $N \in \mathbb{N}$, such that $\frac{a-1}{N} < \epsilon$, so that $x_n < \epsilon$ for all $n \geq N$. Thus, $x_n \to 0$ or $a^{1/n} \to 1$. On the other hand if $0 < a < 1$, then, as proved above, $\frac{1}{a^{1/n}} = \left(\frac{1}{a}\right)^{1/n} \to 1$. So, by Lemma 2.2(c), $a^{1/n} \to 1$.

(b) Let us assume first that $a > 1$. Let $q_n \to q \in \mathbb{Q}$, where $\{q_n\}$ is a sequence of rational numbers. Let $\epsilon > 0$ be given. By (a), choose $N_1 \in \mathbb{N}$, such that $|a^{1/n} - 1| < \frac{\epsilon}{a^q}$ and $|a^{-1/n} - 1| < \frac{\epsilon}{a^q}$ for all $n \geq N_1$. Since $q_n \to q$, choose $N_2 \in \mathbb{N}$, such that $|q_n - q| < \frac{1}{N_1}$ for all $n \geq N_2$. Define $N = \max\{N_1, N_2\}$. Using these inequalities for all $n \geq N$ and the fact that $a > 1$, we have $-\frac{\epsilon}{a^q} < \left(a^{-1/N_1} - 1\right) < \left(a^{q_n - q} - 1\right) < \left(a^{1/N_1} - 1\right) < \frac{\epsilon}{a^q}$, which gives, $\left|a^{q_n-q} - 1\right| < \frac{\epsilon}{a^q}$. So, we have the following for all $n \geq N$

$$|a^{q_n} - a^q| = a^q|a^{q_n-q} - 1| < a^q\frac{\epsilon}{a^q} = \epsilon, \tag{3.4}$$

which proves continuity of f at q for the case $a > 1$.

For $a = 1$, $f(q) = 1$, which being the constant map is continuous.

For $0 < a < 1$, $\frac{1}{a} > 1$, so that the map defined by $q \mapsto \frac{1}{a^q}$ is continuous. Thus, $q_n \to q$ in $\mathbb{Q}$ implies $\frac{1}{a^{q_n}} \to \frac{1}{a^q}$, which by Lemma 2.2(c) gives $a^{q_n} \to a^q$. So, in this case also, f is continuous.

(c) We will prove the case $a > 1$. Let $\epsilon > 0$ be given. Choose a monotonic sequence $\{p_n\}$ of rational numbers, with $p_n \in \left(x - \frac{1}{n}, x - \frac{1}{n+1}\right)$ for all $n \in \mathbb{N}$. Then $|p_n - x| \leq \frac{1}{n(n+1)} < \frac{1}{n}$ for all $n \in \mathbb{N}$, which shows that $p_n \to x$. Moreover, the sequence $\{a^{p_n}\}$ is strictly increasing and bounded above by a^q, where q is a rational number satisfying $x \leq q$. By Theorem 2.3, the sequence $\{a^{p_n}\}$ is convergent. Now define $s_n = q_n - p_n$, so that $s_n \to 0$. By (b), $a^{s_n} \to a^0 = 1$. By Lemma 2.2(b), the sequence $\{a^{s_n} a^{p_n}\}$ converges, which is precisely the sequence $\{a^{q_n}\}$. □

Definition 3.1 (Real exponentiation). For $a > 0$, and $x \in \mathbb{R}$, let $\{q_n\}$ is a sequence of rational numbers, such that $q_n \to x$. Define $a^x = \lim_{n\to\infty} a^{q_n}$. The map defined by $x \mapsto a^x$ is called real exponentiation.

Note that the above definition does not depend upon the sequence $\{q_n\}$ chosen as if $\{q'_n\}$ is another sequence of rational numbers converging to q, then $s_n = (q_n - q'_n) \to 0$ so that $\{a^{s_n}\} \to 1 \Rightarrow \left\{\frac{a^{q^n}}{a^{q'_n}}\right\} \to 1$.

Theorem 3.7. *Let $g : \mathbb{R} \to \mathbb{R}$ have the property that for a given $r \in \mathbb{R}$, any sequence $\{q_n\}$ of rational numbers with $q_n \to r$ implies $g(q_n) \to g(r)$, then the function g is continuous at r.*

Proof. Let $\epsilon > 0$ be given. Let $\{r_n\}$ be a sequence of real numbers converging to r. For each fixed n, let $\{q_{n,m}\}$ be a sequence of rational numbers converging to r_n. So, we can choose $M_n \in \mathbb{N}$, such that $|r_n - q_{n,m}| < \frac{1}{n}$ and $|g(r_n) - g(q_{n,m})| < \frac{1}{n}$ for all $m \geq M_n$. Now $|r - q_{n,M_n}| \leq |r - r_n| + |r_n - q_{n,M_n}| < |r - r_n| + \frac{1}{n}$ for all n, which shows that $\lim_{n\to\infty} q_{n,M_n} = r$. Now by hypothesis, $g(q_{n,M_n}) \to g(r)$. Thus, we have $|g(r_n) - g(r)| \leq |g(r_n) - g(q_{n,M_n})| + |g(q_{n,M_n}) - g(r)| < \left(\frac{1}{n} + |g(q_{n,M_n}) - g(r)|\right)$ for all n, which proves that $g(r_n) \to g(r)$. So, g is continuous at r. □

Corollary 3.8. *For fixed $a \in \mathbb{R}_+$, the function defined by $x \mapsto a^x$ for all $x \in \mathbb{R}$ is continuous.*

Proof. Follows from Theorem 3.7 and definition of real exponentiation. □

For $a > 0$ and $x, y \in \mathbb{R}$, if $p_n \to x$, $q_n \to y$, where $\{p_n\}$ and $\{q_n\}$ are sequences of rational numbers, then, by definition, $a^{p_n} \to a^x$ and $a^{q_n} \to a^y$. By Lemma 2.2(a), $(p_n + q_n) \to (x + y)$, so that $a^{p_n+q_n} \to a^{x+y}$. So, $a^{x+y} = \lim_{n\to\infty} a^{p_n+q_n} = \lim_{n\to\infty}(a^{p_n} a^{q_n})$. By Lemma 2.2, $\lim_{n\to\infty}(a^{p_n} a^{q_n}) = \left(\lim_{n\to\infty} a^{p_n}\right)\left(\lim_{n\to\infty} a^{q_n}\right) = a^x a^y$. So, $a^{x+y} = a^x a^y$.

Let $s_{m,n} = a^{p_m q_n}$, for all $m, n \in \mathbb{N}$. By Corollary 3.4, $a^{p_m} \to a^x$ implies $s_{m,n} = (a^{p_m})^{q_n} \to (a^x)^{q_n}$ as $m \to \infty$. Also, by Corollary 3.8, $s_{m,n} = (a^{q_n})^{p_m} \to (a^{q_n})^x$ as $m \to \infty$. But $s_{m,n} = a^{q_n p_m} \to a^{q_n x}$. By uniqueness of convergence, we have $(a^{q_n})^x = (a^x)^{q_n} = a^{q_n x}$. So, by Corollary 3.8, $a^{x q_n} = (a^x)^{q_n} \to (a^x)^y$. Applying Corollary 3.8 again, $(x q_n) \to (xy)$ implies $a^{x q_n} \to a^{xy}$. By uniqueness of the convergence, $(a^x)^y = a^{xy}$.

Finally, if $b > 0$ is another real number then $(ab)^{p_n} = a^{p_n} b^{p_n}$. By Corollary 3.8, $(ab)^{p_n} \to (ab)^x$, while $a^{p_n} \to a^x$ and $b^{p_n} \to b^x$. Now by Lemma 2.2(b) $(a^{p_n} b^{p_n}) \to (a^x b^x)$. By uniqueness of convergence $(ab)^x = a^x b^x$.

Theorem 3.9. *For $a > 0$, $\lim_{k\to\infty} k(\sqrt[k]{a} - 1)$ exists.*

Proof. If $a = 1$, the sequence $\{k(\sqrt[k]{a} - 1)\}$ is the constant sequence $\{0\}$ which always converges. So, let $a \neq 1$.

In view of Theorem 3.5, we have $k(\sqrt[k]{a} - 1) = \frac{a^{1/k} - 1}{1/k} = g_k(a)$. Note that $g_k(a) = (a - 1) f_{1/k}(a)$, where $f_{1/k}$ is as in (3.3), so that the sequence $\{g_k(a)\}$ is monotonic. If $a > 1$, apply Theorem 3.5 for $x = a$, $y = 1$, $s = 1/k, k > 1$, so that $\frac{1}{k} a^{\frac{1}{k}-1}(a - 1) < (a - 1)\frac{f_{1/k}(a)}{k} < \frac{1}{k}(a - 1)$, which gives $a^{\frac{1}{k}-1}(a - 1) < g_k(a) < (a - 1)$.

On the other hand if $0 < a < 1$, apply Theorem 3.5 for $x = 1$, $y = a$, $s = 1/k, k > 1$, so that $\frac{1}{k}(1 - a) < -(a - 1)\frac{f_{1/k}(a)}{k} < \frac{1}{k} a^{\frac{1}{k}-1}(1 - a)$, which gives $a^{\frac{1}{k}-1}(a - 1) < (a - 1) f_{1/k}(a) < (a - 1)$.

So, for all real positive $a \neq 1$ and $k > 1$, $\sqrt[k]{a}(1 - \frac{1}{a}) < g_k(a) < (a-1)$. Now if $k > 1$ then $\sqrt[k]{a} > 1$ for $a > 1$, and $0 < \sqrt[k]{a} < 1$ for $0 < a < 1$. So, for $a > 1$, $(1 - \frac{1}{a}) < \sqrt[k]{a}(1 - \frac{1}{a})$ and for $0 < a < 1$, $(1-a) < \frac{(1-a)}{\sqrt[k]{a}}$ or $\sqrt[k]{a}(1-a) < (1-a)$ or $\sqrt[k]{a}(1-\frac{1}{a}) > (1-\frac{1}{a})$. Hence, we have $g_k(a) > (1 - \frac{1}{a})$ for all $k > 1$. Thus, the sequence $\{g_k(a)\}$ is monotonically decreasing and is bounded below, which by Theorem 2.3 converges. □

Definition 3.2 (Logarithmic function). Let $\log : \mathbb{R}_+ \to \mathbb{R}$, such that

$$\log a = \lim_{k\to\infty} k(\sqrt[k]{a} - 1). \tag{3.5}$$

The number $\log a$ is called the logarithm of a, and the function log is called the logarithmic function.

From the proof of Theorem 3.9, observe that for any $a \in \mathbb{R}_+$, and $k \in \mathbb{N}$, we have the following important inequalities

$$\left(1 - \frac{1}{a}\right) \leq \log a \leq f_{1/k}(a) \leq (a-1), \tag{3.6}$$

as $\log a = \inf_k\{k(\sqrt[k]{a} - 1)\}$.

Example 3.10. *We may use* (3.6) *to show that*

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1. \tag{3.7}$$

So, let $\epsilon > 0$ *be given. Let* $1 + x = a$, *so that* $x \to 0$ *implies* $a \to 1$. *Consider* $|\frac{1}{a} - 1| = \frac{|a-1|}{a} < \epsilon$ *for* $0 < |a-1| < \delta = a\epsilon$. *We also have* $\lim_{x\to 0} \frac{\log(1+x)}{x} = \lim_{a\to 1} \frac{\log a}{a-1}$. *From* (3.6), *we have* $\left(1 - \frac{1}{a}\right) \leq \log a \leq (a-1)$. *If* $a > 1$, *then* $\frac{1}{a} \leq \frac{\log a}{a-1} \leq 1$. *If* $0 < a < 1$, *then* $\frac{1}{a} \geq \frac{\log a}{a-1} \geq 1$. *In either case, we have* $|\frac{\log a}{a-1} - 1| \leq |\frac{1}{a} - 1| < \frac{a\epsilon}{a} = \epsilon$ *for* $0 < |a-1| < \delta = a\epsilon$, *and the assertion follows.*

Theorem 3.11. *The logarithmic function has the following properties.*

(a) $\log(ab) = \log a + \log b$.
(b) $\log a > 0$ *for* $a > 1$ *and* $\log a < 0$ *for* $0 < a < 1$.
(c) $\log a = 0$ *if and only if* $a = 1$.
(d) log *is strictly increasing.*
(e) log *is injective.*
(f) log *is continuous.*
(g) for any $x \in \mathbb{R}$ *and* $a > 0$, $\log(a^x) = x \log a$.
(h) log *is surjective.*

Proof. (a) We will make use of Lemma 2.2 in the following calculations

$$\begin{aligned}
\log a + \log b &= \lim_{k\to\infty} k(\sqrt[k]{a} - 1) + \lim_{k\to\infty} k(\sqrt[k]{b} - 1) \\
&= \lim_{k\to\infty} \frac{1}{\sqrt[k]{b}}\Big(k(\sqrt[k]{ab} - 1) + k(\sqrt[k]{b} - 1)^2\Big) \\
&= \lim_{k\to\infty} \frac{1}{\sqrt[k]{b}} \lim_{k\to\infty} k(\sqrt[k]{ab} - 1) + \lim_{k\to\infty} \frac{1}{\sqrt[k]{b}} \lim_{k\to\infty} k(\sqrt[k]{b} - 1) \lim_{k\to\infty} (\sqrt[k]{b} - 1) \\
&= 1 \times \log(ab) + 1 \times \log b \times (1 - 1) = \log ab.
\end{aligned}$$

(b) Follows from (3.6).

(c) If $a = 1$, by (a) $\log 1 + \log 1 = \log(1)$, which gives $\log 1 = 0$. The converse follows from (b).

(d) If $0 < a < b$, then $b/a > 1$. By (a) and (c), $0 = \log 1 = \log(aa^{-1}) = \log a + \log(a^{-1})$, so that $\log(a^{-1}) = -\log a$. Now by (b), $\log(b/a) > 0$ or $\log b - \log a > 0$, that is, $\log a < \log b$, which proves that the logarithmic function is strictly increasing.

(e) Follows from (a) and (c) or by (d).

(f) Let $\epsilon > 0$ be given. Let $x_n \to a$ in $\mathbb{R}_+$. By Lemma 2.2(a) and (c), $\frac{x_n}{a} \to 1$ and $\frac{a}{x_n} \to 1$. So we can choose $N_1, N_2 \in \mathbb{N}$, such that $\left|\frac{x_n}{a} - 1\right| < \epsilon$ for all $n \geq N_1$, and $\left|\frac{a}{x_n} - 1\right| < \epsilon$ for all $n \geq N_2$. Define $N = \max\{N_1, N_2\}$. From the aforementioned inequalities, we have for all $n \geq N$, $\left(\frac{x_n}{a} - 1\right) < \epsilon$ and $-\epsilon < \left(1 - \frac{a}{x_n}\right)$. Using these in (3.6) for 'a' $= \frac{x_n}{a} > 0$, we have

$$-\epsilon < \left(1 - \frac{a}{x_n}\right) \leq \log\left(\frac{x_n}{a}\right) \leq \left(\frac{x_n}{a} - 1\right) < \epsilon, \tag{3.8}$$

for all $n \geq N$, which gives $|\log(x_n/a)| < \epsilon$ for all $n \geq N$, that is, $\log(x_n/a) \to 0$ or $(\log x_n - \log a) \to 0$ or $\log x_n \to \log a$. So, the logarithmic function is continuous at a.

(g) First let $x \in \mathbb{Z}$. If $x = 0$ then $a^0 = 1$ and $\log a^0 = 0 \log a = 0$ is satisfied. If $x \in \mathbb{N}$, by (a), we have $\log(a^x) = x \log a$. Also, as $x \log a + x \log(1/a) = \log(a^x a^{-x}) = \log 1 = 0$, we have $\log(a^{-x}) = -x \log a$. So, $\log a^x = x \log a$ for all $x \in \mathbb{Z}$.

Now let $x = p/q \in \mathbb{Q}$, and let $a^{p/q} = b$, that is, $a^p = b^q$. Then $\log(a^p) = \log(b^q)$, which gives $p \log a = q \log b$ or $\log b = \frac{p}{q} \log a$. So, $\log a^x = x \log a$ holds for all $x \in \mathbb{Q}$.

Finally, let $x \in \mathbb{R}$. Let $\{r_n\}$ be a sequence of rational numbers converging to x. By definition, $a^{r_n} \to a^x$. By (f) along with the continuity of real exponentiation, $\log(a^{r_n}) \to \log(a^x)$ or $r_n \log a \to \log(a^x)$. Also, by Lemma 2.2(a), $(r_n \log a) \to (x \log a)$. So, by uniqueness of convergence, $\log(a^x) = x \log a$ as desired.

(h) Let $x \in \mathbb{R}$. We want to find a $y \in \mathbb{R}_+$, such that $x = \log y = \inf_k\{k(\sqrt[k]{y} - 1)\}$, which in turn implies that $x \leq k(\sqrt[k]{y} - 1)$ or $\left(1 + \frac{x}{k}\right) \leq \sqrt[k]{y}$. So, whenever $\left(1 + \frac{x}{k}\right) > 0$, we seek y, such that $\left(1 + \frac{x}{k}\right)^k \leq y$. To find y, we proceed as follows. If $x = 0$, by (c) we have $\log y = 0$ for $y = 1$. So, let $x \neq 0$. For convenience, we define

$$u_k = \left(1 + \frac{x}{k}\right)^k, \ k \in \mathbb{N}. \tag{3.9}$$

Choose $N \in \mathbb{N}$, such that $\left(1 + x/k\right) > 0$ for all $k \geq N$. If $x < 0$, write $x = -t$, where $t > 0$. Whenever, $\left(1 + \frac{x}{k}\right) = \left(1 - \frac{t}{k}\right) > 0$, we have the following

$$\begin{aligned}
\left(1 + \frac{x}{k+1}\right)^{k+1} &= \left(1 - \frac{t}{k+1}\right)^{k+1} = \left(1 - \frac{t}{k} + \frac{t}{k(k+1)}\right)^{k+1} \\
&= \left(1 - \frac{t}{k}\right)^{k+1} + \frac{t}{k}\left(1 - \frac{t}{k}\right)^k + \sum_{i=2}^{k+1} \binom{k+1}{i} \left(1 - \frac{t}{k}\right)^{k+1-i} \frac{t^i}{k^i(k+1)^i} \\
&= \left(1 - \frac{t}{k}\right)^k + \sum_{i=2}^{k+1} \binom{k+1}{i} \left(1 - \frac{t}{k}\right)^{k+1-i} \frac{t^i}{k^i(k+1)^i} \\
&> \left(1 - \frac{t}{k}\right)^k = \left(1 + \frac{x}{k}\right)^k,
\end{aligned} \tag{3.10}$$

since each term in the summation is positive. On the other hand for $x > 0$, we also have

$$\begin{aligned}\Big(1+\frac{x}{k+1}\Big)^{k+1} &= \sum_{m=0}^{k+1}\frac{\binom{k+1}{m}}{(k+1)^m}x^m = \sum_{m=0}^{k+1}\prod_{i=0}^{m-1}\Big(1-\frac{i}{k+1}\Big)\frac{x^m}{m!} \\ &> \sum_{m=0}^{k+1}\prod_{i=0}^{m-1}\Big(1-\frac{i}{k}\Big)\frac{t^m}{m!} > \Big(1+\frac{x}{k}\Big)^k.\end{aligned} \tag{3.11}$$

From (3.10)-(3.11), we have $u_k < u_{k+1}$ for all $x \in \mathbb{R}$ so that the sequence $\{u_k\}_{k\geq N}$ is strictly increasing. By (d), the sequence $\{\log(u_k)\}_{k\geq N}$ is also strictly increasing.

Take $M \in \mathbb{N}$, such that $|x|/M < 1$, so that we have $|u_k| \leq \sum_{m=0}^{M-1}\frac{|x|^m}{m!}+\sum_{m=M}^{\infty}\frac{|x|^m}{m!} \leq \sum_{m=0}^{M-1}\frac{|x|^m}{m!}+ |x|^M\sum_{m=0}^{\infty}\frac{|x|^m}{M^m} = \sum_{m=0}^{M-1}\frac{x^m}{m!}+\frac{|x|^M}{1-|x|/M}$, which proves that $\{u_k\}$ is bounded. By, Theorem 2.3, the sequence $\{u_k\}$ converges. So, let $u_k \to y$. Note that $y > 0$. By (f), $\log u_k \to \log y$.

Also, using (3.6) for $a = u_k$, $k \geq N$, we have $k(1-\frac{1}{1+x/k}) \leq \log u_k \leq k(1+x/k-1) = x$, which gives $0 \leq x - \log u_k \leq x(1-\frac{1}{1+x/k}) \to 0$ as $k \to \infty$. Thus, $\log u_k \to x$. By uniqueness of convergence, we have $x = \log y$ as desired. □

Theorem 3.11 establishes that the logarithmic function is continuous and bijective. The proof of the fact that $\lim_{k\to\infty} u_k$ exists, motivates the next.

Definition 3.3 (**Exponential function).** Let $\exp : \mathbb{R} \to \mathbb{R}_+$, such that

$$\exp(x) = \lim_{k\to\infty}\Big(1+\frac{x}{k}\Big)^k. \tag{3.12}$$

The function exp is called the exponential function.

Theorem 3.12. $\log^{-1} = \exp$.

Proof. Let $\epsilon > 0$ be given, and let $x > 0$. We can choose $N \in \mathbb{N}$, such that $\Big(1+\frac{\log x}{k}\Big) > 0$, $\Big(k(\sqrt[k]{x}-1)-\log x\Big) < \frac{\epsilon}{(1+\epsilon)x}$, and $|x^{-\frac{1}{k}}-1| < \epsilon$ for all $n \geq N$. If we let $p_k = \Big(1+\frac{\log x}{k}\Big)^k$, by Theorem 3.11, $\log x \leq k(\sqrt[k]{x}-1)$ implies $p_k \leq \Big(1+\frac{k(\sqrt[k]{x}-1)}{k}\Big)^k = x$, so that $0 < p_k \leq x$ for all $k \geq N$. On applying the first inequality of Theorem 3.5 for $r = k \geq N$, $\sqrt[k]{x} > \big(1+\frac{\log x}{k}\big) > 0$, we get

$$0 \leq \big((\sqrt[k]{x})^k - p_k\big) \leq x^{1-\frac{1}{k}}(k(\sqrt[k]{x}-1)-\log x),$$

from which we have

$0 \leq (x-p_k) \leq x^{1-\frac{1}{k}}(k(\sqrt[k]{x}-1)-\log x) < x(1+\epsilon)\frac{\epsilon}{(1+\epsilon)x} = \epsilon$ for all $n \geq N$. So, $p_k \to x$, that is, $\exp(\log x) = x$.

Now for any $y \in \mathbb{R}$, by Theorem 3.11(h), choose $x \in \mathbb{R}_+$, such that $\log x = y$. As $u_k \to y = \exp(x)$, by Theorem 3.11(f), $\log u_k \to \log y = \log(\exp(x))$. But in the proof of Theorem 3.11(h), we have $\log u_k \to x$. So, by uniqueness of convergence, $\log(\exp(y)) = y$. □

Corollary 3.13. $\exp(x+y) = \exp(x)\exp(y)$ *for all* $x, y \in \mathbb{R}$.

Proof. Let $\exp(x) = x_1$ and $\exp y = y_1$. Then $x = \log x_1$ and $y = \log y_1$. So, $x+y = \log x_1+\log y_1 = \log(x_1y_1)$, which implies $\exp(x+y) = \exp(\log(x_1y_1)) = x_1y_1 = \exp(x)\exp(y)$. □

Corollary 3.14. *For any* $x \in \mathbb{R}$, $\exp(x) = (\exp(1))^x$.

Proof. First observe that for any real number $a > 0$,
$a^x = \exp(\log(a^x)) = \exp(x \log a) = \lim_{k\to\infty}\left(1 + \frac{x\log a}{k}\right)^k$, which on taking $a = \exp(1)$ gives $(\exp(1))^x = \lim_{k\to\infty}\left(1 + \frac{x\log\exp(1)}{k}\right)^k = \lim_{k\to\infty}\left(1 + \frac{x}{k}\right)^k = \exp(x)$, as desired. □

Theorem 3.15. *For any $a \in \mathbb{R}_+$, $\lim_{h\to 0}\left(\frac{a^h-1}{h}\right)$ exists and is equal to $\log a$.*

Proof. Let $g(h) = \frac{a^h-1}{h}$ for $h \neq 0$.

We first show that g is strictly increasing on $(0, 1]$. So, let $h_1, h_2 \in (0, 1]$, where $h_1 < h_2$. Choose two sequences of rational numbers $\{s_n\}$ and $\{t_n\}$, such that $\{s_n\}$ is monotonically decreasing and $s_n \to h_1$; $\{t_n\}$ is monotonically increasing with $s_1 \le t_1$ and $t_n \to h_2$. One example of such $\{s_n\}$ and $\{t_n\}$ could be

$$s_n \in \Big\{\mathbb{Q} \cap \Big(h_1 + \frac{\lambda}{n+2}, h_1 + \frac{\lambda}{n+1}\Big)\Big\}; t_n \in \Big\{\mathbb{Q} \cap \Big(h_2 - \frac{\lambda}{n+1}, h_2 - \frac{\lambda}{n+2}\Big)\Big\},$$

where $\lambda = (h_2 - h_1)$, and $n \in \mathbb{N}$. Clearly, $s_n < t_n$ for all n, $s_n \to h_1$ and $t_n \to h_2$, where $h_1 = \inf_n\{s_n\}$ and $h_2 = \sup\{t_n\}$. By continuity of g, $g(s_n) \to g(h_1)$ and $g(t_n) \to g(h_2)$. Since $g(q) = (a-1)f_q(a)$ for all $q \in \mathbb{Q}_+$, where f_q is as in (3.3), which is found to be strictly increasing, we have $g(s_n) < g(t_n)$, and $\{g(s_n)\}$ and $\{g(t_n)\}$ are strictly decreasing and strictly increasing sequences, respectively. Therefore, $g(s_n) \to \inf_n\{g(s_n)\} = g(h_1)$ and $g(t_n) \to \sup_n\{g(t_n)\} = g(h_2)$ so that $g(h_1) \le g(s_n) < g(t_n) \le g(h_2)$, from which we get $g(h_1) < g(h_2)$.

So, for any $h \in (0, 1)$, if we choose $k \in \mathbb{N}$ satisfying $0 < \frac{1}{k} < h < 1$ then $\log a \le k(\sqrt[k]{a} - 1) = g(1/k) < g(h) < g(1) = (a-1)$. So, g is bounded on $(0, 1]$.

We have proved that g is a strictly monotone, continuous, and bounded function on the interval $(0, 1]$. Now we prove that $\lim_{h\to 0^+} g(h)$ exists and is equal to $\alpha = \inf_{0<h<1}\{g(h)\}$. So, let $\epsilon > 0$ be given. Then $(\alpha + \epsilon)$ is not a lower bound of the set $\{g(h) \mid 0 < h < 1\}$. So, there is a $\delta \in (0, 1)$, such that $g(\delta) < (\alpha + \epsilon)$. Then for all $0 < h < \delta$, we have $0 < (g(h) - \alpha) \le (g(\delta) - \alpha) < \epsilon$. So, $\lim_{h\to 0^+} g(h) = \alpha$. Now for $0 < h < 1$, $g(-h) = a^{-h}g(+h)$, where $\lim_{h\to 0} a^{-h} = a^0 = 1$. So, $\lim_{h\to 0^-} = 1 \times \lim_{h\to 0^+} g(h) = \alpha$. Thus, $\lim_{h\to 0} g(h) = \alpha$.

Finally, let $\epsilon' > 0$ be given. Let $\delta' > 0$ be a real number such that $0 < |h| < \delta'$ implies $0 \le |g(h) - \alpha| < \epsilon'$. Then for each positive integer $k > 1$, where $\frac{1}{k} < \delta'$, we have $0 \le g(1/k) - \alpha < \epsilon'$, which proves that $g(1/k) \to \alpha$. Since $g(1/k) \to \log a$, by uniqueness of convergence, $\alpha = \log a$. □

Corollary 3.16. $\lim_{h\to 0}\left(\frac{\exp(h)-1}{h}\right) = 1$.

Proof. By Corollary 3.14 and Theorem 3.15, we have

$\lim_{h\to 0}\left(\frac{\exp(h)-1}{h}\right) = \lim_{h\to 0} \frac{(\exp(1))^h-1}{h} = \log(\exp(1)) = 1.$ □

Theorem 3.17. *If $a > 0$ and $r \in \mathbb{R}$, then $\lim_{x\to a} \frac{x^r-a^r}{x-a} = ra^{r-1}$.*

Proof. Let $\epsilon > 0$ be given. We will prove the result in the following cases.

If $r \in \mathbb{N}$, we have $\frac{x^r-a^r}{x-a} = \frac{(x-a)(\sum_{m=1}^r x^{r-m}a^{m-1})}{(x-a)} = \sum_{m=1}^r x^{r-m}a^{m-1}$. So, $\lim_{x\to a} \frac{x^r-a^r}{x-a} = \lim_{x\to a} \sum_{m=1}^r x^{r-m}a^{m-1} = \sum_{m=1}^r a^{r-m}a^{m-1} = ra^{r-1}$, where the last step follows from Theorem 2.7 since a polynomial function is continuous by Theorem 3.3(a) and Theorem 2.8(a).

Now we will extend the result to all integers. The case $r = 0$ is trivial. We only need to establish the case of negative integers. So, let $r \in \mathbb{N}$, and consider $\frac{x^{-r}-a^{-r}}{x-a} = \frac{x^r-a^r}{x-a} \times \frac{-1}{(xa)^r}$. Thus, $\lim_{x\to a} \frac{x^{-r}-a^{-r}}{x-a} = \lim_{x\to a} \frac{x^r-a^r}{x-a} \lim_{x\to a} \frac{-1}{(xa)^r} = ra^{r-1} \times \frac{-1}{a^{2r}} = -ra^{-r-1}$ as desired.

Now we will extend the result to all rational numbers. So, let $r = \frac{p}{q}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Let $x^{\frac{1}{q}} = u$ and $a^{\frac{1}{q}} = v$ so that $x = u^q$ and $a = v^q$. Note that $x \to a$ implies $u \to v$. Then we have $\lim_{x\to a} \frac{x^r-a^r}{x-a} = \lim_{u\to v} \frac{u^p-v^p}{u^q-v^q} = \lim_{u\to v} \frac{u^p-v^p}{u-v} \lim_{u\to v} \frac{u-v}{u^q-v^q} = \frac{pv^{p-1}}{qv^{q-1}} = rv^{q(r-1)} = ra^{r-1}$.

Finally, let $r \in \mathbb{R}$. Let $\{q_n\}$ be a sequence of rational numbers, such that $q_n \to r$. Then $\lim_{x\to a} \frac{x^{q_n}-a^{q_n}}{x-a} = q_n a^{q_n-1}$. So, for a given $\epsilon > 0$, there is a $\delta > 0$, such that $0 < |x-a| < \delta$ implies $\left|\frac{x^{q_n}-a^{q_n}}{x-a} - q_n a^{q_n-1}\right| < \frac{\epsilon}{3}$. Since $q_n \to r$, by continuity of exponentiation, $\left(\frac{x^{q_n}-a^{q_n}}{x-a}\right) \to \left(\frac{x^r-a^r}{x-a}\right)$. Also, $q_n a^{q_n-1} \to ra^{r-1}$. So, choose $N \in \mathbb{N}$, such that $\left|\frac{x^{q_n}-a^{q_n}}{x-a} - \frac{x^r-a^r}{x-a}\right| < \frac{\epsilon}{3}$ and $|q_n a^{q_n-1} - ra^{r-1}| < \frac{\epsilon}{3}$ for all $n \geq N$. Now consider the following for $0 < |x-a| < \delta$.

$$\begin{aligned}\left|ra^{r-1} - \frac{x^r-a^r}{x-a}\right| &\leq \left|ra^{r-1} - q_N a^{q_N-1}\right| + \left|q_N a^{q_N-1} - \frac{x^{q_N}-a^{q_N}}{x-a}\right| + \left|\frac{x^{q_N}-a^{q_N}}{x-a} - \frac{x^r-a^r}{x-a}\right| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,\end{aligned}$$

which completes the proof. □

4 Concluding Remarks

The present exposition is aimed at deriving some of the standard results on limit. In doing so, we have used the concept of convergence to define continuity of functions. These results on limits are useful in obtaining derivatives of real valued functions defined on subsets of $\mathbb{R}$. The derivative of such a function f at a point a of its domain is defined by

$$\frac{d}{dx} f(x) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a}, \tag{4.1}$$

provided that the limit exists.

Within the framework of the present setup, one can prove the basic formulas on derivatives. For example, using Corollary 3.16, we can show that

$$\frac{d}{dx} \exp(x) = \exp(x) \lim_{h\to 0} \left(\frac{\exp(h) - 1}{h}\right) = \exp(x) \times 1 = \exp(x), \tag{4.2}$$

which avoids the use of infinite series expansion of the exponential function. Similarly, Theorem 3.17 can be used to prove that

$$\frac{d}{dx} x^r = rx^{r-1} \tag{4.3}$$

for all real r and $x > 0$, which avoids the use of Taylor's expansion.

The continuity of the functions defined on subsets of $\mathbb{R}$ plays vital role in understanding the real exponentiation, which in the present exposition has been defined via convergence. Then the connection between the limit and continuity is established, which facilitates many calculations on limit. For instance, Theorem 2.9 can be used to show that

$$\lim_{x\to 0} (1+x)^{\frac{1}{x}} = \lim_{x\to 0} \exp\left\{\frac{\log(1+x)}{x}\right\} = \exp\left\{\lim_{x\to 0} \frac{\log(1+x)}{x}\right\} = \exp(1), \tag{4.4}$$

where the last step is obtained using (3.7).

Similarly, Theorem 3.12 can be used to show that for a positive real number $a \neq 1$, the map defined by $x \mapsto a^x$ for all real x is bijective with continuous inverse, since the exponential function is bijective and $a^x = \exp(x \log a)$. Moreover, the inverse map is defined by $y \mapsto \left(\frac{\log y}{\log a}\right)$ for all positive real y, since $a^{\frac{\log y}{\log a}} = \exp\left(\frac{\log y}{\log a} \log a\right) = \exp(\log y) = y$ and $\frac{\log(a^x)}{\log a} = x$ for all real x. The inverse map so obtained defines the logarithm of y with respect to the base a denoted by $\log_a y$, that is, if $y = a^x$ then we write $\log_a y = x = \frac{\log y}{\log a}$.

Finally, to prove that $\frac{d}{dx} \sin x = \cos x$, one needs to show that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1. \tag{4.5}$$

This requires a discussion on complex exponential function, and limit and continuity of functions defined on subsets of $\mathbb{R}^2$, which can be done via desiring componentwise existence of these concepts. After doing that, one can establish the existence of $\lim_{k\to\infty} \left(1 + \frac{\iota x}{k}\right)^k$, and define this limit to be the complex exponential $\exp(\iota x)$. Then the Corollary 3.16 can be extended to the complex exponential, which after defining $\sin x = \dfrac{\exp(\iota x) - \exp(-\iota x)}{2\iota}$ will establish (4.5). We recommend the reader to go through another different approach of obtaining (4.5) in [6] via purely non-geometric definitions of the trigonometric functions.

We would like the reader to consult the excellent texts by Bloch [7], Stillwell [8], and Kumar and Kumaresan [9] on the present lines.

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Competing Interests

Authors have declared that no competing interests exist.

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