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On the Notes of Quasi-Boundary Value Method for Solving Cauchy-Dirichlet Problem of the Helmholtz Equation

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Authors' contributions

This work was carried out in collaboration by all authors. Author BB designed the study and introduced the algorithm for the corrected class of ill-posed problem. Author FOB wrote the first draft of the manuscript. Author SKA optimized the method and guided the experimental results. Author EO-F corrected the grammatical errors in the draft. All authors read and approved the final manuscript.

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Abstract

The Cauchy-Dirichlet problem of the Helmholtz equation yields unstable solution, which when solved with the Quasi-Boundary Value Method (Q-BVM) for a regularization parameter $\alpha = 0$. At this point of regularization parameter, the solution of the Helmholtz equation with both Cauchy and Dirichlet boundary conditions is unstable when solved with the Q-BVM. Thus, the quasi-boundary value method is insufficient and inefficient for regularizing ill-posed Helmholtz equation with both Cauchy and Dirichlet boundary conditions. In this paper, we introduce an expression $\frac{1}{(1+\alpha^2)}$, $\alpha \in \mathbf{R}$, where α is the regularization parameter, which is multiplied by $w(x, 1)$ and then added to the Cauchy and Dirichlet boundary conditions of the Helmholtz equation. This regularization parameter overcomes the shortcomings in the Q-BVM to account for the stability at $\alpha = 0$ and extend it to the rest of values of $\mathbf{R}$.

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1 Introduction

In recent times, ill-posed problems have drawn the attention of mathematicians and scientists in general, in the global world. These problems cause either distortion in the signal processes or breaks in the vibrating strings. An ill-posed problem comes as a result of imposed boundary conditions on the Helmholtz equation. Thus, the stability of the solution of Helmholtz equation depends on the type of the boundary conditions imposed on the equation.

Definition 1.1. *A Laplace-type operator occuring in the Helmholtz equation* $A : \Omega \subset H \to H$*, where* $A = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ *and* H *is the Hilbert space, is called Hölder-continuous on* Ω *with constant* γ *and exponent* p *if there exists* $\gamma \geq 0$ *and* $p \in (0, 1]$ *such that*

$$\|Aw_1(x, y) - Aw_2(x, y)\| \leq \gamma \|w_1(x, y) - w_2(x, y)\|^p, \ \ \forall \, w_1, w_2 \in \Omega \subset H.$$

If $p = 1$*, then* $A(.)$ *is called uniformly Lipschitz-continuous on* Ω*, with Lipschitz constant* γ*. See* [1].

Since the pioneering work by [2] who introduced Tikhonov Regularization Method (TRM) a lot of studies have been done to regularize the Helmholtz equation with Cauchy and Dirichlet boundary conditions in the Hilbert space. The authors in [3], introduced boundary knot method for regularizing inhomogeneous Helmholtz equation. In [4], the authors obtained regularized solution for ill-posed Helmholtz equation with discontinuities at the points in its domain. The authors in [5] introduced the Convex Regularization Method (CRM) for solving Cauchy problem of the Helmholtz equation in the Hilbert space. They obtained the continuity of Laplace-type operator $A^{-1}(.)$ occuring in the Helmholtz equation. They showed that the solution of the Helmholtz equation by using CRM was better as compared to Iterative Regularization Method (IRM) for different values of regularization parameter.

One of the methods of regularization which gives vivid picture of stability of solution of the Helmholtz equation is the Q-BVM. In this method, it is assumed that the inverse Laplace-type operator $A^{-1}(.)$ occuring in the Helmholtz equation exists from a Hilbert space to a subHilbert space $\Omega \subset H$ but not continuous. When the continuity of $A^{-1}(.)$ is restored, then the $A(.)$ is well-posed in the Hadamard sense. For example, see papers by [6, 7]. In order to achieve the continuity of $A^{-1}(.)$, several algorithms have been introduced, see [8]. The authors in [9, 10], added a product of a regularization parameter $\alpha \in \mathbf{R}^+$ and a boundary condition at non-fixed spatial variable to the boundary deflection in the Cauchy data imposed on the Helmholtz equation. In [11], the author regularized Helmholtz equation by subtracting $\alpha w'(0)$ instead of adding $\alpha w(0)$ to the initial condition $w(T)$ in heat conduction problem. In [12], the author implemented the Q-BVM to regularize a linear elliptic equation. Using the Q-BVM, authors in [13] regularize unbounded Dirichlet problem of the Poisson equation in $L^2(\mathbf{R})$. In [14] and [15], authors regularized nonlinear and linear heat equation, respectively.

One of such studies which is recent and of interest was given by [16]. In their method, the regularization parameter α is restricted for only positive real numbers, but not for other values of real numbers. The small range of values of α used in [16] and others, do not give the liberty to explore the full characteristics of the (well-posed) Helmholtz equation in the Hilbert space.

In this paper, we extend the range of values of the regularization parameter from $\mathbf{R}^+$ to $\mathbf{R}$ to account for the fully characteristics (stability) of the Helmholtz equation with both Cauchy and Dirichlet boundary conditions in the Hilbert space. Thus, this method of regularization of the Helmholtz equation with both Cauchy and Dirichlet boundary conditions accounts for stability at $\alpha = 0$ and the rest of values of $\alpha \in \mathbf{R}$. At $\alpha = 0$ is a crucial point in regularizing Cauchy-Dirichlet problem of the Helmholtz equation in the Hilbert space. In order to do this, we introduce an expression $\frac{1}{(1+\alpha^2)}$, $\alpha \in \mathbf{R}$, which is multiplied by $w(x,1)$ and then added to the Cauchy and Dirichlet boundary conditions of the Helmholtz equation.

This paper is organized as follows. In section 1, we provide the introduction to Q-BVM for solving ill-posed Helmholtz equation with both Cauchy and Dirichlet boundary conditions. In section 2, we show that the Helmholtz equation with both Cauchy and Dirichlet boundary conditions is ill-posed. Section 3 deals with modified Q-BVM. Thus, we show that the Helmholtz equation with both Cauchy and Dirichlet boundary conditions is well-posed in the sense of Hadamard. In section 4, we discuss our results and compare it to existing methods of regularization. Section 5 contains the conclusion of the paper.

2 Solving An Ill-posed Cauchy-Dirichlet Problem of the Helmholtz Equation Using Q-BVM

In this section, we show that the solution of the Helmholtz equation with imposed both Cauchy and Dirichlet boundary conditions is unstable. Thus, the small perturbations in these boundary conditions result in large changes in the solution of the Helmholtz equation.

Lemma 2.1 (Data Compactibility Condition). *A Laplace-type operator occuring in the Helmholtz equation $A : \Omega \subset H \to H$, denotes a bounded region in $\mathbf{R}^2$ having a smooth boundary $\partial\Omega$. The problem below:*

$$\Delta w(x,y) = 0 \quad in \;\; \Omega$$
$$\frac{\partial w(x,0)}{\partial y} = 0 \quad on \;\; \partial\Omega$$

has no solution unless the data functions on the right hand sides of the above two equations satisfy compactibility condition

$$\int_{\Omega} 0 d\Omega = \int_{\partial\Omega} 0 dx.$$

See [17].

The Helmholtz equation together with both the Cauchy and the Dirichlet boundary conditions is ill-posed in the sense of Hadamard. The Cauchy-Dirichlet problem of the Helmholtz equation is as follows:

$$\begin{aligned} \frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2} &+ k^2 w(x,y) = 0, \quad 0 \le x \le 2\pi, \;\; 0 \le y \le 1 \\ w(0,y) = w(2\pi,y) &= 0, \qquad 0 \le y \le 1 \\ w(x,0) &= \sin(nx), \qquad 0 \le x \le 2\pi \\ \frac{\partial w(x,0)}{\partial y} &= 0, \qquad 0 \le x \le 2\pi, \end{aligned} \tag{1}$$

where k is the wave number.

By the method of separation of variables, we obtain the solution of equation (1) in the cartesian coordinates as:

$$w(x,y) = \sum_{n=0}^{\infty} \cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}y\right) \sin(\frac{nx}{2}) \sin(nx) \qquad (2)$$

We show that the solution that appears in equation (2) is unstable with respect to the small changes in the boundary conditions. In equation (1), we choose $x_1 = 0$ in boundary condition $w(x,0) = \sin(nx)$ and the corresponding solution is obtained as follows:

$$w_1(x,y) = 0$$

Again, we perturb from $x_1 = 0$ to $x_2 = \delta$, where $0 < \delta \ll \frac{\pi}{24}$ and the corresponding solution is obtain as:

$$w(x,y) = \sum_{n=1,3}^{\infty} 4\sin(n\delta) \cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}y\right) \sin(\frac{nx}{2}) \sin(nx)$$

The change in the boundary condition is observed as:

$$\begin{aligned} \lim_{n\to\infty} |w(x_1,0) - w(x_2,0)| &= \lim_{n\to\infty} |0 - \sin(n\delta)| \\ \lim_{n\to\infty} |w(x_1,0) - w(x_2,0)| &\to 1 \ \ as \ \ n \to \infty. \end{aligned}$$

The above result implies that there is a small change in the boundary condition.

The corresponding change in the solution $w(x,y)$ is

$$\begin{aligned} \lim_{n\to\infty} |w_1(x,y) - w_2(x,y)| &= \lim_{n\to\infty} |0 - \sum_{n=1}^{\infty} 4\sin(n\delta) \cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}y\right) \sin(\frac{nx}{2}) \sin(nx)| \\ \lim_{n\to\infty} |w_1(x,y) - w_2(x,y)| &\leq \lim_{n\to\infty} \sum_{n=1}^{\infty} 4e^{(\sqrt{((\frac{n}{2})^2 - k^2)}y)} \\ \lim_{n\to\infty} |w_1(.,1) - w_2(.,1)| &\leq \lim_{n\to\infty} \sum_{n=1}^{\infty} 4e^{\sqrt{((\frac{n}{2})^2 - k^2)}} \end{aligned}$$

Thus,

$$\lim_{n\to\infty} |w_1(.,1) - w_2(.,1)| \to \infty \ \ as \ \ n \to \infty.$$

This implies that a small change in the boundary condition $w(x,0)$ from $x_1 = 0$ to $x_2 = \delta$ results in a large change in solution. Thus, the solution (2) to equation (1) is unstable in the Hilbert space. The equation (1) violates the third condition of well-posedness. Hence, equation (1) is ill-posed in the sense of Hadamard.

However, we showed that Q-BVM cannot be used to restore stability of solution of Hadamard equation at $\alpha = 0$. Thus, the continuity of $A^{-1}(.)$ in the Helmholtz equation is not restored. The discontinuity of $A^{-1}(.)$ disturbs numerical computation of solution of the Helmholtz equation in the Hilbert space. To see this problem of discontinuity of $A^{-1}(.)$ in the Helmholtz equation, let us consider the following:

$$\begin{aligned} \frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2} &+ k^2 w(x,y) = 0, \quad 0 \le x \le 2\pi, \quad 0 \le y \le 1 \\ w(0,y) = w(2\pi,y) &= 0, \qquad 0 \le y \le 1 \\ w(x,0) + \alpha w(x,1) &= \sin(nx), \qquad 0 \le x \le 2\pi \\ \frac{\partial w(x,0)}{\partial y} &= 0, \qquad 0 \le x \le 2\pi, \end{aligned} \tag{3}$$

with the solution

$$w(x,y) = \frac{\sum_{n=0}^{\infty} \cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}y\right) \sin(\frac{nx}{2}) \sin(nx)}{\left(1 + \alpha \cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}\right)\right)}. \tag{4}$$

We can see that when $\alpha = 0$, then equation (3) and the function which appears in equation (4) become equations (1) and (2), respectively. Hence, by Q-BVM, equation (3) is (unstable) ill-posed in the sense of Hadamard.

3 Main Results

The solution of the Cauchy-Dirichlet problem of the Helmholtz equation is unstable, which when solved with Q-BVM for regularization parameter $\alpha = 0$. At this point, the boundary conditions in equation (3) and the solution that appears in equation (4) become equations (1) and (2), respectively. Thus, Helmholtz equation together with Cauchy and Dirichlet boundary conditions is (unstable) ill-posed in the sense of Hadamard.

Definition 3.1 (First Vanishing Theorem). *A Laplace-type operator occuring in the Helmholtz equation $A : \Omega \subset H \to H$, be continuous function on the closure of the domain $\bar{\Omega}$. Assume that $A \ge 0$ in the $\bar{\Omega}$ and that $\int\int_\Omega Aw(x,y)dxdy = 0$. Then $Aw(x,y)$ is identically zero. See* [18].

In this paper, we modify the Q-BVM to account for the stability at $\alpha = 0$. By the dint of this method, we introduce an expression $\frac{1}{(1+\alpha^2)}$, where, $\alpha \in \mathbf{R}$ is the regularization parameter, in $w(x,0)$ of Cauchy-Dirichlet problem of the Helmholtz equation. The product of the regularization term $\frac{1}{(1+\alpha^2)}$ and $w(x,1)$ is added to $w(x,0)$ in the both Cauchy and Dirichlet problems of the Helmholtz equation presented in equation (1). Hence, the name modified Q-BVM. This method suppresses the high frequency component in the solution of the Helmholtz equation, which accounts for the rapid decay of the solution in the Hilbert space.

$$\begin{aligned} \frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2} &+ k^2 w(x,y) = 0, \quad 0 \le x \le 2\pi, \quad 0 \le y \le 1 \\ w(0,y) = w(2\pi,y) &= 0, \qquad 0 \le y \le 1 \\ w(x,0) + \frac{1}{(1+\alpha^2)} w(x,1) &= \sin(nx), \qquad 0 \le x \le 2\pi \\ \frac{\partial w(x,0)}{\partial y} &= 0, \qquad 0 \le x \le 2\pi, \end{aligned} \tag{5}$$

By the method of separation of variables, we obtain (unique) solution of the equation (5) in the cartesian coordinates as follows:

$$w(x,y) = \frac{\sum_{n=0}^{\infty} \cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}y\right) \sin(\frac{nx}{2}) \sin(nx)}{\left(1 + \frac{1}{(1+\alpha^2)} \cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}\right)\right)} \tag{6}$$

We show that equation (5) is well-posed. Firstly, we see that the Neumann boundary condition vanishes on the boundary of the domain, the function that appears in equation (2) is a solution to equation (1).

Secondly, we prove that equation (5) has only one solution.

Proof: By contradiction, we suppose that equation (5) has two different smooth solutions denoted by $u(x, y)$ and $v(x, y)$. Also, let

$$w(x, y) = u(x, y) - v(x, y).$$

Multiplying both sides of equation (5) by $w(x, y)$ and integrating over $([0, 2\pi] \times [0, 1])$, we obtain

$$\int_0^1 \int_0^{2\pi} w(x, y)\Delta w(x, y)dxdy + k^2 \int_0^1 \int_0^{2\pi} |w(x, y)|^2 dxdy = 0 \tag{7}$$

By the first vanishing theorem, the right hand side is zero. Applying the Green's first identity to the first term on the left hand side of equation (7), we obtain

$$\int_0^1 \int_0^{2\pi} w(x, y)\Delta w(x, y)dxdy = -\int_0^1 \int_0^{2\pi} |\nabla w(x, y)|^2 dxdy \tag{8}$$

Substituting equation (8) into equation (7) yields

$$-\int_0^1 \int_0^{2\pi} |\nabla w(x, y)|^2 dxdy + k^2 \int_0^1 \int_0^{2\pi} |w(x, y)|^2 dxdy = 0. \tag{9}$$

For equation (9) to hold, we restrict

$$\int_0^1 \int_0^{2\pi} |w(x, y)|^2 dxdy = 0$$

By the first vanishing theorem, we obtain

$$\Rightarrow w(x, y) = 0.$$

Also, we can see from equation (9) that

$$\begin{aligned} -\int_0^1 \int_0^{2\pi} |\nabla w(x, y)|^2 dxdy &= 0 \\ \Rightarrow |\nabla w(x, y)| &= 0 \\ \Rightarrow w(x, y) &= 0. \end{aligned}$$

This implies that $w(x, y)$ is a smooth function, constant in the domain and vanishes on the boundary of the domain. Thus,

$$u(x, y) = v(x, y)$$

Hence, the function that appears in equation (6) is the only solution to equation (5).

Last but not least, we show that the small changes in the both Cauchy and Dirichlet boundary conditions result in small changes in the solution that appears in equation (6).

In equation (5), we perturb from $x_1 = 0$ to $x_2 = \delta$, where $0 < \delta \ll \frac{\pi}{24}$, and the change in boundary conditions is observed as:

$$\lim_{n\to\infty} |w(x_1, 0) - w(x_2, 0)| \quad \rightarrow \quad 1 \;\; as \;\; n \rightarrow \infty.$$

The above result implies that there is a small change in the boundary condition.

The corresponding change in the solution $w(x, y)$ is

$$\begin{aligned}
\lim_{n\to\infty} |w_1(x,y) - w_2(x,y)| &\leq \lim_{n\to\infty} |\sum_{n=0}^{\infty} \frac{\cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}y\right)}{\left(1 + \frac{1}{(1+\alpha^2)}\cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}\right)\right)}| \\
\lim_{n\to\infty} |w_1(.,1) - w_2(.,1)| &= \lim_{n\to\infty} |\sum_{n=0}^{\infty} \frac{\cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}\right)}{\left(1 + \frac{1}{(1+\alpha^2)}\cosh\left(\sqrt{((\frac{n}{2})^2 - k^2)}\right)\right)}| \\
\lim_{n\to\infty} |w_1(.,1) - w_2(.,1)| &\leq \lim_{n\to\infty} \sum_{n=0}^{\infty} \frac{e^{\left(\sqrt{((\frac{n}{2})^2 - k^2)}\right)}}{\left(1 + \frac{1}{(1+\alpha^2)}e^{\left(\sqrt{((\frac{n}{2})^2 - k^2)}\right)}\right)}
\end{aligned}$$

At $\alpha = 0$, we obtain

$$\lim_{n\to\infty} |w_1(.,1) - w_2(.,1)| = \lim_{n\to\infty} \sum_{n=0}^{\infty} \frac{e^{\left(\sqrt{((\frac{n}{2})^2 - k^2)}\right)}}{\left(1 + e^{\left(\sqrt{((\frac{n}{2})^2 - k^2)}\right)}\right)}$$

We can see that the denominator grows faster than the numerator, we obtain

$$\lim_{n\to\infty} |w_1(.,1) - w_2(.,1)| \to 1 \ \ as \ \ n \to \infty.$$

This implies that the solution which appears in equation (6) is stable to the small changes in the boundary conditions. Thus, the continuity of $A^{-1}(.)$ is restored. Hence, equation (5) together with regularized Cauchy and Dirichlet boundary conditions is well-posed in the sense of Hadamard in the Hilbert space.

4 Results and Discussion

In this section, we the present the main findings of the work. In figure 1, we display the result by equation (3) for regularization parameter $\alpha = 0$, in two dimensions. In figure 4, we display the solution of equation (5) given by a red solid graph and that of equation (3) by a blue solid graph. At $\alpha = 0$, we observed that the equations (1) and (3) yield the same solution. We can see from figures 1, 2 and 3 that there is a gradual increase in the solution given by equation (5), whereas the solution of equation (3) increases sharply. This implies that equation (5) yields a stable solution. On the other hand, the solution of equation (3) is unstable. For instance, a small change in y results in large change in the solution $w(\pi, y)$; indicated by a blue solid graph (see figure 3). But, the red solid graph gives consistent, accurate and stable results. Thus, a small change in y leads to a small change in the solution $w(\pi, y)$. In addition, we display the solution of equation (5) for $\alpha = 0.1$ in two dimensions and different values of α in one dimension in figures 4 and 5, respectively. We observed that the solution of equation (5) becomes consistent and stable as α becomes small.

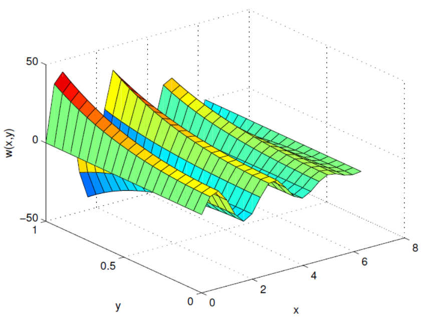

Fig. 1. Solution by the Q-BVM at $\alpha = 0$

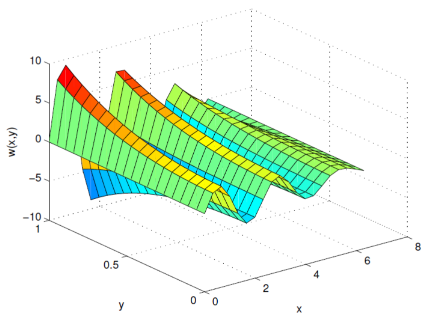

Fig. 2. Solution by the MQ-BVM at $\alpha = 0$

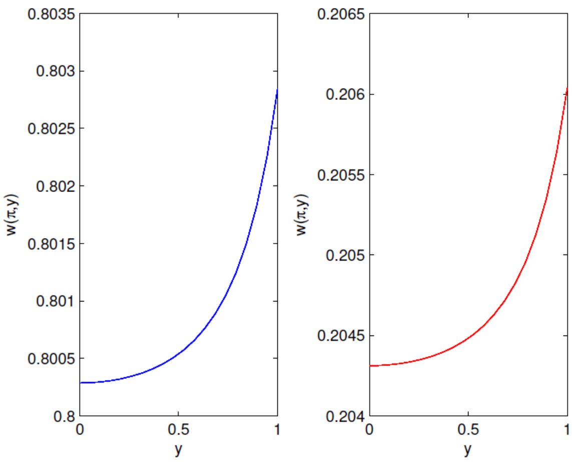

Fig. 3. Comparison of solutions by the Q-BVM and by the MQ-BVM at $\alpha = 0$ **in one dimension**

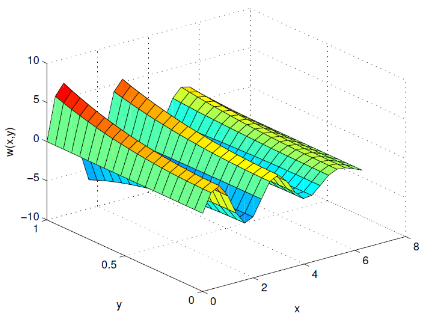

Fig. 4. Solution by the MQ-BVM at $\alpha = 0.1$

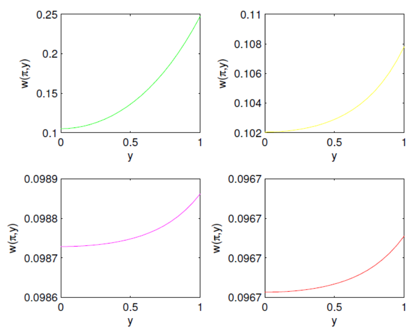

Fig. 5. Solutions by the MQ-BVM at different values of α

5 Conclusion

We observed that the solution of both Cauchy and Dirichlet problems of the Helmholtz equation is stabilized by the MQ-BVM for regularization parameter $\alpha = 0$. On the other hand, the Q-BVM cannot stabilize the solution of the Helmholtz equation together with both Cauchy and Dirichlet boundary conditions at $\alpha = 0$. Also, solution of Helmholtz equation together with both Cauchy and Dirichlet boundary conditions by the classical method is the same as one by the Q-BVM at $\alpha = 0$, which in turn, is ill-posed.

Last but not least, the MQ-BVM provides accurate, consistent and stable solution of the Helmholtz equation together with both Cauchy and Dirichlet boundary conditions when the regularization parameter becomes small as compared to other methods of regularization.

Competing Interests

Authors have declared that no competing interests exist.

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