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Length Functions and HNN Groups

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The sole author designed, analyzed and interpreted and prepared the manuscript.

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ABSTRACT

The concept of length functions on groups is first introduced by Lyndon. This is used to give direct proofs of many other results in combinatorial group theory. Two important sets called M and N satisfying some certain axioms of length functions are considered. Finally investigations of the nature and the structures of the sets M and N in relation to the elements of HNN group were carried out.

Keywords: Archimedean elements; associated subgroups; conjugate elements; Coset representative; H. N. N extension; isomorphism; length functions; normal form; reduced word.

1. INTRODUCTION

In this paper we define a construction given by G. Higman, B. H. Neumann and H. Neumann in 1949. This construction is called HNN extension.

We define a length function on HNN extensions to get some further results concerning the structure of HNN extensions, factor groups and some predefined important parts of this group. However, we formulate a normal form theorem for HNN extensions and consider reduced forms of the elements of this group in order to investigate the structure of this extension in details. The background of Length Functions is based on the issues raised in, [1-6].

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2. LENGTH FUNCTIONS

2.1 Definition: A length function | | on a group G, is a function giving each element x of G a real number $|x|$, such that for all $x, y, z \in G$, the following axioms are satisfied.

$A1'$ $|e| = 0$, e is the identity elements of G .

$A2$ $|x^{-1}| = |x|$

$A4$ $d(x,y) < d(y,z) \Longrightarrow d(x,y) = d(x,z)$,

where $d(x,y) = \frac{1}{2}\,(\,|x| + |y| - |xy^{-1}|$

Lyndon in [1] showed that $A4$ is equivalent to $d(x,y) \geq \min\{\,d(y,z), d(x,z)\}$ and to

$d(y,z), d(x,z) \geq m \Longrightarrow d(x,z) \geq m$.

$A1'$, $A2$ and $A4$ imply that: $|x| \geq d(x,y) = d(y,x) \geq 0$.

Assuming, A2 and A4 only, it is easy to show that:

i. $d(x,y) \geq |e|$
ii. $|x| \geq |e|$
iii. $d(x,y) \leq |x| - \frac{1}{2}|e|$, see [5]

The axiom A3 states that: $d(x,y) \geq 0$ is deducible from $A1'$, $A2$. Also, $A1'$ is a weaker version of the following axiom: $A1$: $|x| = 0$ if and only if $x = 1$ in G.

Lyndon [1] showed that if G is any group with length function and x, y and z are elements in G, then the following properties will hold.

2.2 Proposition: $d(xy,y) + d(x,y^{-1}) = |y|$

2.3 Proposition: $d(x,y^{-1}) + d(y,z^{-1}) \leq |y|$ implies that $|x\,y\,z| \leq |x| - |y| + |z|$

2.4 Proposition: $d(x,y^{-1}) + d(y,z^{-1}) \leq |y|$ implies that $d(xy,z^{-1}) = d(y,z^{-1})$

2.5 Proposition: $d(x,y) + d(x^{-1},y^{-1}) \geq |x| = |y|$ implies that $|(xy^{-1})^2| \leq |xy^{-1}|)$

It follows from proposition 2.2 that for any $x, y \in G$, $d\,(x,y) = |y| - d\,(x\,y^{-1}, y^{-1}) \leq |y|$ by A3.

Since $d(x,y) = d\,(y\,,x)$, we get: $d\,(x\,,y) \leq \min\{|x|\,,|y|\}$.

A5 states that: $d\,(x\,,y) + d\,(x^{-1}\,,y^{-1}) > |x| = y \Rightarrow x = y$

2.6 Definition: A non-trivial element g of a group G is called Non-Archimedean if $|g^2| \leq |g|$

2.7 Definition: Let G be a group with length function. An element $x \neq 1$ in G is called Archimedean if $|x| \leq |x^2|$.

The following Axioms and results have added by Lyndon and others

$A0$ $x \neq 1 \Longrightarrow |x| < |x^2|$

$C0$ $d(x\,,y\,)$ is always an integer

$C1$ $x \neq 1$, $|x^2| \leq |x|$ *implies* $|x|$ is odd.

$C2$ $For\ no\ x\ is\ |x^2| = |x| + 1$

$C3$ $if\ |x|\ is\ odd\ then\ |x^2| \geq |x|$

$C1'$ $if\ |x|\ is\ even\ and\ |x| \neq 0\,, then\ |x^2| > |x|$

$N0$ $|x^2| \leq |x|\ implies\ x^2 = 1\ \ is\ x = x^{-1}$

$N1^*$ $G\ is\ general\ by\ \{x \in G :\ |x| \leq 1\}$

2.8 Definition: The set of all Non-Archimedean elements in G will be denoted by N, which is given by: $N = \{x \in G : |x^2| \le |x|\}$

Lyndon also introduced the following set in [1]: $M = \{xy \in G : |xy| + |yx| < 2|x| = 2|y|\}$, and showed that $M \subseteq N$. The nature of the elements of M and N is investigated in the next section.

3. HNN EXTENSION

We now introduce an important group constructed by G. Higman, B.H. Neumann and H. Neumann. The details of this construction are given in, [7], [8] and [9].

3.1 Definition: Let G be a group, I be an index set and $\{A_i : i \in I\}$, $\{B_i : i \in I\}$ be families of subgroups of G with a family $\{\emptyset_i : i \in I\}$ of maps such that, each $\emptyset_i : A_i \to B_i$ is an isomorphism. Then the H.N.N extension with base G_i and stable letters t_i , $i \in I$ and associated subgroups A_i and B_i, $i \in I$ is the group

$G^* = < G, t_i ; rel\ G,\ t_i^{-1} a_i t_i = \emptyset_i(a_i),\ a_i \in A_i >$, where $\langle G, rel\ G \rangle$ is a presentation of G.

To formulate a normal from theorem for H.N.N extensions, we shall consider the following:

Any element of G^* is equal to a product $g_0\, t_{i_1}^{\varepsilon_1} g_1 \dots\ t_{i_n}^{\varepsilon_n} g_n, n \ge 0,\ \varepsilon_i = \pm 1$

Note: Throughout this section g_i will denote an element of G.

3.2 Definition: A sequence $g_0\, t_{i_1}^{\varepsilon_1} g_1 \dots\ t_{i_n}^{\varepsilon_n} g_n, n \ge 0,$ $\varepsilon_i = \pm 1$ is said to be reduced if there is no consecutive subsequence $t_i^{-1} g_i\, t_i$ with $g_i \in A_i$, $or\ t_i g_i t_i^{-1}$ with $g_i \in B_i$ if w is a word in $G \cup \{t_i\} \cup \{t_i^{-1}\}$. Then we can get $t_i - reduction$ of w corresponding to the relations of G^* as follows:

1) Replace a sub word of the from $t_i^{-1} g_i\, t_i$, by $\emptyset_i(g_i)$ whenever $g_i \in A_i$
2) Replace a sub word of the from $t_i\ g_i\, t_i$, by $\emptyset_i(g_i)$ whenever $g_i \in B_i$

By carrying out all possible $t_i - reduction$ we get a reduced word defining the same element of G^*.

The products of the elements in two distinct reduced sequences may be equal in G^*. To get normal forms, once again we consider the coset representatives as follows:

Choose for each i a set of representatives of the right cosets of $A_i\ in\ G$ and a set of representatives of the right cosets of $B_i\ in\ G$. We shall assume that 1 is the representative of both cosets A_i and B_i.

3.3 Definition: Given the sets of right coset representatives of A_i and B_i in G, then a normal form in G^* is a sequence of the form $g_0\, t_{i_1}^{\varepsilon_1} g_1 \dots\ t_{i_n}^{\varepsilon_n} g_n, n \ge 0\ \varepsilon_r = \pm 1$, where

i) g_0 is an arbitrary element of G, except that $g_0 \ne 1$ if n=0
ii) $if\ \varepsilon_r = -1$ then g_r is a representative of a coset of $A_{i_r}\ in\ G$
iii) $if\ \varepsilon_r = +1$ then g_r is a representative of a coset of $B_{i_r}\ in\ G$ and
iv) There is no subsequence $t^{\varepsilon}\, 1\, t^{-\varepsilon}$ where $\varepsilon = \pm 1$.

Because of the relations $t_i^{-1} a_i t_i = \emptyset_i(a_i)$ of G^*, we can replace $t_i^{-1} a_i$ by $\emptyset_i(a_i) t_i^{-1}$ without changing the corresponding element of G. Similarly we can replace $t_i b_i$ by $\emptyset_i^{-1}(b_i) t_i^{-1}$, by working from right to left. We can show that every element of G^* is equal to a product $g_0\, t_{i_1}^{\varepsilon_1} \dots\ t_{i_n}^{\varepsilon_n} g_n$ where $g_0\, t_{i_1}^{\varepsilon_1} \dots\ t_{i_n}^{\varepsilon_n} g_n$ is a normal form.

3.4 Theorem (Normal Form Theorem): Let $G^* = < G, t_i ; rel\ G,\ t_i^{-1} a_i t_i = \emptyset_i(a_i),\ a_i \in A_i,$ $i \in I >$ be an H.N.N. extension. Then every element of G^* has a unique representation as a product $g_0\, t_{i_1}^{\varepsilon_1} \dots\ t_{i_n}^{\varepsilon_n} g_n$, where $g_0\, t_{i_1}^{\varepsilon_1} \dots\ t_{i_n}^{\varepsilon_n} g_n$ is in a normal form.

Proof: See [8]

3.5 Theorem (Higman, Neumann, Neumann): Let $G^* = < G, t_i ; rel\ G,\ t_i^{-1} a_i t_i = \emptyset_i(a_i),\ a_i \in A_i,\ i \in I >$ be H.N.N. extension, then the group G is embedded in G^* by the map$g \to g$.

3.6 Theorem (Britton's Lemma): If $g_0\, t_{i_1}^{\varepsilon_1} \dots\ t_{i_n}^{\varepsilon_n} g_n = 1$ in G^* where ≥ 1 , then $g_0\, t_{i_1}^{\varepsilon_1} \dots\ t_{i_n}^{\varepsilon_n} g_n$ is not reduced.

Theorems 3.5 and 3.6 are equivalent to theorem 2.4 (proofs are given in [1] and [9]).

3.7 Lemma: Let $G^* = < G, t_i ; rel\ G,\ t_i^{-1} a_i t_i = \emptyset_i(a_i),\ a_i \in A_i,\ i \in I >$ be H.N.N. extension and $u = g_0\ t_{i_1}^{\varepsilon_1}\ \dots\ t_{i_n}^{\varepsilon_n}\, g_n$, $v = h_0\ t_{i_1}^{\delta_1}\ \dots\ t_{i_m}^{\varepsilon_m}\, h_m$ be reduced words such that u = v in G^*. Then $m = n$ and $\varepsilon_i = \delta_i\,, for\ \ i = 1, \dots, n$.

Proof: Since u = v in G^* , then $1 = g_0\, t_{i_1}^{\varepsilon_1}\ \dots\ t_{i_n}^{\varepsilon_n} g_n h_m^{-1} t^{-\delta_m} \dots t^{-\delta_1} h_0^{-1}$.

Since u and v are reduced, the only way the indicated sequence can fail to be reduced is that when $\varepsilon_n = \delta_m$ and $g_n h_m^{-1}$ is in the appropriate sub-group A_i or B_i. Successive t-reductions will result in, each $\varepsilon_i = \delta_i$ and m = n.

The normal form theorem 3.4 for H.N.N. extension allows us to assign a well-defined length to each element of these extensions.

3.8 Definition: Let $G^* =< G, t_i\,; rel\ G\,,\ t_i^{-1} a_i t_i = \emptyset_i(a_i)\,,\ a_i\ \in\ A_i\,,\ i\ \in\ I >$ be an H.N.N. extension. Define the length of an element g $in\ G^*$ by:

$|g| = n\,,\ if\ g = g_0\ t_{i_1}^{\varepsilon_1}\ \dots\ t_{i_n}^{\varepsilon_n}\, g_n\,, n \geq 0$, is in a reduced form, where $\varepsilon_1 = \pm 1$.

The following Theorems generalize some results in [10] and [11].

3.9 Theorem: Let $G^* =< G, t_i\,; rel\ G\,,\ t_i^{-1} a_i t_i = \emptyset_i(a_i)\,,\ a_i\ \in\ A_i\,,\ i\ \in\ I >$ be H.N.N. extension and $|g| = n\,, if\ g = g_0\ t_{i_1}^{\varepsilon_1}\ \dots\ t_{i_n}^{\varepsilon_n}\, g_n\,, n \geq 0$, be in a reduced form, where $\varepsilon_1 = \pm 1$. Then | | is a length function on G^*.

Proof

$A1'\ \ |1| = 0$

$A2\ \ |g| = |g^{-1}|,\ \ g\ \in G^*$ is obvious as g^{-1} will be reduced if g is reduced.

Let $g,\ h,\ k\ \in G^*$

Suppose $d\,(g\,,h)\,,\ d\,(h\,,t\,) \geq s$

Let $g = x_1 t_1^{\varepsilon_1} x_1^{-1}\ \dots .\, x_n t_n^{\varepsilon_n} x_n^{-1}$, $|g| = n \geq 1$ and $h = y_1 t_1^{\varepsilon_1} y_1^{-1}\ \dots .\, y_m t_m^{\varepsilon_m} y_m^{-1}$, $|h| = m \geq 1$ be in reduced forms

$gh^{-1} = x_1 t_1^{\varepsilon_1} x_1^{-1} \dots\ x_n t_n^{\varepsilon_n} x_n^{-1} t^{-1} y_m^{-1} \dots\ y_1 t_1^{\varepsilon_1} y_1^{-1}$

$x_n^{-1} y_m = 1\ \in G$. Then $t_m^{-1}\, 1\, t_m^{-1} = 1\ \in G$.

Suppose $gh^{-1} = x_1 t_1^{\varepsilon_1} x_1^{-1} \dots\ x_{n-s} a_s y_{m-s}^{-1}\ \ \dots .\, y_1^{-1}$

Let $k = z_1 t_1^{\varepsilon_1} z_1^{-1} \dots\ z_n t_n^{\varepsilon_n} z_n^{-1}$ and $gk^{-1} = gh^{-1}\, hgk^{-1}$

$hk^{-1} = y_1 t_1^{\varepsilon_1} y_1^{-1} \dots\ y_{n-s} b_s\, z_{u-s}^{-1} t_{m1}^{\varepsilon_{m-1}}\, z_{u-s} \dots\ z_1^{-1}$

Therefore $gh^{-1} = x_1 t_1^{\varepsilon_1} x_1^{-1} \dots\ x_{n-s} a_s y_{m-s}^{-1}\ \ \dots .\, y_1^{-1} k_u^{-1} t_u^{\varepsilon_u} k_u \dots z_1^{-1}$

As $d(g,h)$ and $(h,k) \geq s$. Then

$gh^{-1} = x_1 t_1^{\varepsilon_1} x_1^{-1} \dots\ a_{s+1} b_{s+1} z_{u-s} t_{n-1}^{\varepsilon_{n-1}}\, z_{u-s} \dots\ z_1^{-1}$

$Therefore\ |gh^{-1}| \leq n + u - 2s$, i e $d(g,k) \geq s$

So | | is a length function.

It is proved in [6], that d(g,h) is always an integer, i.e C0 is satisfied in H.N.N extensions.

3.10 Theorem: Let $G^* =< G, t_i\,;\ rel\ G\,,\ t_i^{-1} a_i t_i = \emptyset_i(a_i)\,,\ a_i\ \in\ A_i\,,\ i\ \in\ I >$ be H.N.N extension.

Then the elements of N are the conjugates of the elements of the base G.

Proof: To show that if $g \in N$, then $g = x\,a\,x^{-1}, x \in G^*$ and $a \in G$.

Suppose that $g \in N$ and $g = x_1 t_1^{\varepsilon} x_2 \ \dots\ x_{n-1} t_n^{\varepsilon_n} x_n$ is reduced. i e $|\,g\,| = n$

The result is trivial if n = 0 or 1.

Now if $|\,g^2\,| \le |g|$, then

$$g^2 = x_1 t_1^{\varepsilon_1} x_2 \ \dots\ x_{n-1} t_n^{\varepsilon_n} x_n x_1 t_1^{\varepsilon_1} x_2 \ \dots\ x_{n-1} t_n^{\varepsilon_n} x_n,$$

Where, $|x_s a_s x_{s+1}| \le 2$, if there is no further cancellations.

$$g = \underbrace{\left(x_1 t_1^{\varepsilon_1} x_2 \ \dots\ x_{n-1} t_n^{\varepsilon_n} x_n\right)\left(x_1 t_1^{\varepsilon} x_2 \ \dots\ x_{s-1} t_s^{\varepsilon_s} x_s\right)}\left(x_1 t_1^{\varepsilon_1} x_2 \ \dots\ x_{s-1} t_s^{\varepsilon_s} x_s\ \right)^{-1},$$

Where,

$$a_s = \left(x_1 t_1^{\varepsilon_1} x_2 \ \dots\ x_{n-1} t_n^{\varepsilon} x_n\right)\left(x_1 t_1^{\varepsilon} x_2 \ \dots\ x_{s-1} t_s^{\varepsilon_s} x_s\right)$$

Therefore

$$g = \left(x_1 t_1^{\varepsilon} x_2 \ \dots\ x_{s-1} t_s^{\varepsilon_s} x_s\right) \quad a_s \left(x_1 t_1^{\varepsilon_1} x_2 \ \dots\ x_{s-1} t_s^{\varepsilon_s} x_s\ \right)^{-1}$$

Therefore g is a conjugate of an element a_s of G.

Conversely suppose $g = \left(x_1 t_1^{\varepsilon_1} x_2 \ \dots\ x_{s-1} t_s^{\varepsilon_s} x_s\right) a_s \left(x_1 t_1^{\varepsilon_1} x_2 \ \dots\ x_{s-1} t_s^{\varepsilon_s} x_s\ \right)^{-1}$

If $x_s a_s x_s^{-1} \in G$ then put $x_s a_s x_s^{-1} = a_{s+1}$. This means that $a_s \in G$

If $|a_{s+1}| = 0$, then $|a_{s+1}^2| = 0$, so $a_{s+1} \in G$. If $|a_{s+1}| = 1$ then $a_{s+1}^2 = x_s a_s x_s^{-1}$ where $a_s^2 \in G$
Suppose $x_s a_s x_s^{-1}$ is reduced, I e $|a_{s+1}^2| = 2$

Therefore $a_s \in G$, which is a contradiction. So $x_s a_s x_s^{-1}$ is reduced, ie $\left|a_{s+1}\right| = \left|a_s\right|$

$Therefore, a_{s+1} \in G$

Further, if $x_{s-1} a_{s+1} x_{s-1}^{-1}$ is not reduced, then

$$g = \left(x_1 t_1^{\varepsilon_1} x_2 \dots x_{r-1} t_{r-1}^{\varepsilon_{r-1}} x_r\right) b_r \left(x_1 t_1^{\varepsilon_1} x_2 \dots x_{r-1} t_{r-1}^{\varepsilon_{r-1}} x_r\right)^{-1},$$

Where $b \in G$ and $x_r\, b_r x_r^{-1}$ is reduced

If $b \in G$, then $|g| = 2\,r$, $b^2 \in G$ and $|g^2| = \left|\left(x_1 t_1^{\varepsilon_1} x_2 \dots x_s\right)\left(x_1 t_1^{\varepsilon_1} x_2 \dots x_s\right)^{-1}\right| \le 2r$

So $g \in N$, $b \notin G$, $x_r b$ and $b x_r^{-1}$ are reduced resulting in $|g| = 2r + 1$

Since $|b^2| \le 1$, then $|g^2| = \left|\left(x_1 t_1^{\varepsilon_1} x_2 \dots x_s\right) b_s^2 \left(x_1 t_1^{\varepsilon_1} x_2 \dots x_s\right)^{-1}\right| \le 2r + 1 = |g|$

Therefore, $g \in N$
In case if $b \notin G$ and either $x_r b_r x_r^{-1}$ is not reduced, then $|g| = 2r$

Since $\in N$, then $|b^2| \le |b|$. So, b^2 is not reduced.

Consider $g^2 = (x_1 t_1^{\varepsilon_1} x_2 \dots x_s) b_s^2 (x_1 t_1^{\varepsilon_1} x_2 \dots x_s)^{-1}$ and suppose $x_r b^2 x_r^{-1}$ is reduced, then:

Either $b_r x_r^{-1}$ is not reduced or $x_r b$ is not reduced.

Therefore $x_r b^2 x_r^{-1}$ is not reduced ie $|x_r b^2 x_r^{-1}| \leq 2$

Therefore $|g^2| = 2r = |g|$, $i\, e\; g \in N$.

3.11 Theorem: Let $G^* = < G, t_i \,; rel\; G\,,\; t_i^{-1} a_i t_i = \emptyset_i(a_i)\,,\; a_i \in A_i\,,\; i \in I >$ be H.N.N. extension. Then the elements of N are equivalent if and only if they are conjugates by the same elements of G^*. **(if** $g{\sim}h$ in N then $|gh^{-1}| \leq |g| = |h|$)

Proof Let $g = x_0 t_1^{\varepsilon_1} x_2 \dots x_{n-1} t_n^{\varepsilon_n} x_n$ and $h = y_0 t_1^{\varepsilon_1} y_2 \dots y_{n-1} t_n^{\varepsilon_n} y_n$ be both reduced

1) The result is trivial if $|g| = |h| = 0{,}1$
2) If n > 1

where $\boldsymbol{\varepsilon}_i = \mp 1$.

By theorem 3.10, $g = (x_0 t_1^{\varepsilon_1} \in_1 \dots x_{n-s} t_n^{\varepsilon_n} x_s) a_s (x_0 t_1^{\varepsilon_1} \dots x_s)^{-1}$, where $a_s \in G$ and

$$h = (y_0 t_1^{\varepsilon_1} y_2 \dots y_s) b_s (y_0 t_1^{\varepsilon_1} \dots y_s) \;,\; b_s \in G$$

$$gh^{-1} = (x_0 t_1^{\varepsilon_1} x_1 \dots x_{s-1})(t_{s-1} a_s)\;\; (x_0 t_1^{\varepsilon_1} \dots x_s)^{-1} (y_0 t_1^{\varepsilon_1} y_1 \dots y_s)(b_s^{-1} y_s^{-1})\, (y_0 t_1^{\varepsilon_1} y_1 \dots y_{s-1})^{-1}$$

Since $|gh^{-1}| \leq n$, then $(x_0 t_1^{\varepsilon_1} x_1 \dots x_s)^{-1} (y_1 t_1^{\varepsilon_1} \dots y_s) = G$, then

$$y_0 t_1^{\varepsilon_1} y_1 \dots y_s = (x_0 t_1^{\varepsilon} x_1 \dots x_s) a_s$$

Thus $h = (x_0 t_1^{\varepsilon_1} x_1 \dots x_s)\;\; a_s b_s a_s^{-1} (x_0 t_1^{\varepsilon_1} x_1 \dots x_s)^{-1}$ where $a_s b_s a_s^{-1} = a \in G$

Hence g, h are conjugate of $a \in G$.

Conversely suppose that $g = (x_0 t_1^{\varepsilon_1} x_1 \dots x_r)\;\; a_s (x_0 t_1^{\varepsilon_1} x_1 \dots x_s)^{-1}$, where $a_r \in G$ and

$$h = (y_0 t_1^{\varepsilon_1} y_1 \dots y_r)\;\; b_r (y_0 t_1^{\varepsilon_1} \dots y_r)^{-1}, b_r \in G \text{ where } a{\sim}b$$

Similar argument show that $x_r a_r x_r^{-1}$ is not reduced.

Since $a{\sim}b$, then either $a_r\,, b_r \in G$ then $a_r b_r^{-1} \in G$ and

$$|gh^{-1}| = \left|(x_0 t_1^{\varepsilon_1} x_2 \dots x_r)(x_0 t_1^{\varepsilon_1} x_2 \dots x_r)^{-1}\right| \leq 2r. \text{ So } g{\sim}h.$$

3.12 Theorem: Let $G^* = < G, t_i \,; rel\; G\,,\; t_i^{-1} a_i t_i = \emptyset_i(a_i)\,,\; a_i \in A_i\,,\; i \in I >$ be H.N.N. extension.

Then the elements of M are the conjugates of the elements of the associated subgroups.

Proof: To prove that $g\,, h \in M \rightarrow gh = xax^{-1}$, where $x \in G$.

Let $g = x_0 t_1^{\varepsilon_1} x_1 \dots x_n\,, h = y_0 t_1\, y_1 \dots y_n$ be reduced and suppose

$$|gh\;| + |hg| < 2|h| = |g|, \text{ Then } |g|, |h| \geq 1.$$

$n = 1$ is trivial.

Suppose $|y_1x_1\ | = 0$, then $gh = x_1y_1x_1x_1^{-1}$= conjugate of (y_1x_1)

Similarly if $|y_1x_1\ | = 0$, then hg is conjugate of x_1y_1

So let $n \geq 2$ and let

$$gh = (x_0t_1^{\varepsilon_1}x_2 \dots x_s)a_s(y_{s+1} \dots y_n) \quad , \tag{1}$$

So $s \leq n$ and s is maximum.

Then (1) is reduced in which case, $|gh\ | = 2n - 2s + 1$ or $S_s\ \in G$ and $x_{n-s}a_sy_{s+1}$ is reduced, in which case $|gh\ | = 2n - 2s$, where $a_s = x_{n-s+1} \dots t_n^{\varepsilon_1}x_n\, y_1t_1^{\varepsilon_1}y_1 \dots y_s$

Similarly

$$hg = y_0t_1^{\varepsilon_1}y_1 \dots y_{n-r}b_rx_{r+1}t_{r+2}^{\varepsilon_{r+2}} \dots x_n \tag{2}$$

Then either (2) is reduced so $|hg\ | = 2n - 2r + 1$ or $b_r \in G$ and $y_{n-r}b_rx_{r+1}$ is not reduced so

$|hg\ | = 2n - 2r$, Where $b_r = y_{m-r+1}t_{m-r+1}^{\varepsilon_i} \dots\ y_1\, x_1t_1^{\varepsilon}y_1 \dots x_r$, Then $2n - 2s + 1 + 2n - 2r + 1 < 2n$

$$2n - 2r - 2s + 2 < 0$$

$r + s > n + 1$, Therefore, $r > n - s + 1$ and $s > n - r + 1$

Then $b_{n-s+1}\ \in G$. Since $a_{s-1}\ \in G$, then $b_{n-s+1}a_{s-1}\ \in G$ or gh is a conjugate of an element in G.

4. CONCLUSION

The conclusion is that in any HNN group, the elements of N are conjugates of the base group G and they are conjugate of each other by the same elements. However the elements of M are the conjugates of the associated groups. Moreover, the set M is a subset of N.

COMPETING INTERESTS

Author has declared that no competing interests exist.

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