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Short Communication

Properties of the Euler phi-function on pairs of positive integers (6*x* - 1, 6*x* + 1)

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Let n ≥ 1 be an integer. Define ϕ2(n) to be the number of positive integers x, 1 ≤ x ≤ n, for which both 6x−1 and 6x+1 are relatively prime to 6n. The primary goal of this study is to show that ϕ2 is a multiplicative function, that is, if gcd(m, n) = 1, then ϕ2(mn) = ϕ2(m)ϕ2(n).

Key words: Euler phi-function, multiplicative function.

THE EULER Φ2 FUNCTION

Let $n \geq 1$ be an integer and let $S = \{1, 7, 11, 13, 17, 19, 23, 29\}$, the set of integers which are both less than and relatively prime to 30. In Mothebe and Modise (2016) we define $\phi_3(n)$ to be the number of integers x, $0 \leq x \leq n-1$, for which $\gcd(30n, 30x+i) = 1$ for all $i \in S$. We proved that this function is multiplicative and thereby obtained a formula for its evaluation.

For each $n \in \mathbb{N}$ let $\phi_2(n)$ denote the number of positive integers x, $1 \leq x \leq n$, for which both $6x-1$ and $6x+1$ are relatively prime to $6n$. In this note we draw analogy with our study of ϕ_3 and show that ϕ_2 is multiplicative.

For example if $n = 5 \in S$, then $\phi_2(n) = 3$ since the pairs (11, 13), (17, 19) and (29, 31) are the only ones with components that are relatively prime to 30.

We now proceed to show that we can evaluate $\phi_2(n)$ from the prime factorization of n. Our arguments are based on those used by Burton (2002) to show that the Euler phi-function is multiplicative. We first note:

Theorem 1.1: Let k and s be nonnegative numbers and let p ≥ 5 be a prime number. Then the following hold:

(i) $\phi_2(2^k) = 2^k$.
(ii) $\phi_2(3^s) = 3^s$.
(iii) $\phi_2(p^k) = p^k - 2p^{k-1}$

Proof. (i) and (ii): For all nonnegative integers k and s and x:

$$\gcd(6x-1, 6 \cdot 2^k) = \gcd(6x+1, 6 \cdot 2^k) = 1$$

and

$$\gcd(6x-1, 6 \cdot 3^s) = \gcd(6x+1, 6 \cdot 3^s) = 1.$$

Proof. (iii): Clearly $\gcd(6x-1, 6p^k) = 1$ if and only if p does not divide $6x-1$ and $\gcd(6x+1, 6p^k) = 1$ if and only if p does not divide $6x+1$. There is one integer between 1 and p that satisfies the congruence relation $6x \equiv 1 \pmod p$. Hence there are p^{k-1} integers between 1 and p^k that satisfy $6x \equiv 1 \pmod p$. Similarly there are p^{k-1} integers of the form $6x+1$ between 1and p^k divisible by p. Thus the set $\{(6x-1, 6x+1) \mid 1 \leq x \leq$

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p^k } contains exactly $p^k - 2p^{k-1}$ pairs corresponding to integers x for which both $\gcd(6x - 1, 6p^k) = 1$ and $\gcd(6x + 1, 6p^k) = 1$. Thus $\phi_2(p^k) = p^k - 2p^{k-1}$.

For example $\phi_2(5^2) = 5^2$-2.5=15 and $\phi_2(5)$ =5-2·5^0 =3 as observed earlier.

We recall that:

Definition 1.2: A number theoretic function f is said to be multiplicative if $f(mn) = f(m)f(n)$ whenever $\gcd(m, n) = 1$.

From the proof of Theorem 1.1 it is clear that for all integers k and s: $\phi_2(2^k 3^s) = \phi_2(2^k)\phi_2(3^s)$.

We now show that the function ϕ_2 is multiplicative. This will enable us to obtain a formula for $\phi_2(n)$ based on a factorization of n as a product of primes. We require the following results:

Lemma 1.3: Given integers m, n, gcd(6x − 1, 6mn) =1 if and only if gcd(6x − 1, 6m) = 1 and gcd(6x − 1,6n) =1.

Similarly given integers m, n, $\gcd(6x + 1, 6\,mn) = 1$ if and only if $\gcd(6x + 1, 6m) = 1$ and $\gcd(6x + 1, 6n) = 1$. This is an immediate consequence of the following standard result.

Lemma 1.4: Given integers m, n, k, gcd(k, mn) = 1 if and only if gcd(k, m) = 1 and gcd(k, n) = 1.

We note also the following standard result.

Lemma 1.5: If a = bq + r, then gcd(a, b) = gcd(b, r).

Theorem 1.6: The function ϕ_2 is multiplicative, that is, if gcd(m, n) = 1, then $\phi_2(mn) = \phi_2(m)\phi_2(n)$.

Proof: The result holds if either m or n equals 1. We shall therefore assume neither m nor n equals 1. Arrange the integer pairs $(6x - 1, 6x + 1)$, $1 \le x \le mn$, in an $n \times m$ array as follows:

$$\begin{bmatrix} (5,7) & \cdots & (6m-1, 6m+1) \\ (6(m+1)-1, 6(m+1)+1) & \cdots & (6(2m)-1, 6(2m)+1) \\ \vdots & \vdots & \vdots \\ (6((n-1)m+1)-1, 6((n-1)m+1)+1) & \cdots & (6(mn)-1, 6(mn)+1) \end{bmatrix}$$

we know that $\phi_2(mn)$ is equal to the number of pairs $(6x - 1, 6x + 1)$ in this matrix for which both $6x - 1$ and $6x + 1$ are relatively prime to '6mn'. By virtue of Lemma 1.3 this is the same as the number of pairs $(6x - 1, 6x + 1)$ in the same matrix for which both $6x-1$ and $6x+1$ are relatively prime to each of $6m$ and $6n$.

We first note, by virtue of Lemma 1.5, that $\gcd(6(qmx) - 1, 6m) = \gcd(6x - 1, 6m)$ and likewise $\gcd(6(qm+x)+1, 6m) = \gcd(6x+1, 6m)$. Therefore the pairs $(6(qm + x) - 1, 6(qm + x) + 1)$ in the x^{th} column are both relatively prime to $6m$ if and only if both $6x - 1$ and $6x + 1$ are relatively prime to $6m$.

Therefore only $\phi_2(m)$ columns contain pairs $(6x - 1, 6x + 1)$ both relatively prime to $6m$ and every other pair in the column will constitute of integers both relatively prime to $6m$. The problem now is to show that in each of these $\phi_2(m)$ columns there are exactly $\phi_2(n)$ integer pairs $(6x - 1, 6x + 1)$ that are both relatively prime to $6n$, for then altogether there would be $\phi_2(m)\phi_2(n)$ pairs in the table that are relatively prime to both $6m$ and $6n$.

The entries that are in the x^{th} column (where it is assumed $\gcd(6x-1, 6m) = \gcd(6x + 1, 6m) = 1$) are: $(6x - 1, 6x + 1)$, $(6(m + x) - 1, 6(m + x) + 1)$, . . . , $(6((n-1)m+x)-1, 6((n-1)m+x)+1)$.

There are n pairs in this sequence and for no two pairs: $(6(qm + x) - 1, 6(qm + x) + 1)$, $(6(jm + x) - 1, 6(jm + x) + 1)$ in the sequence do we have:

$$6(qm + x) - 1 \equiv 6(jm + x) - 1 \pmod{n}$$

and

$$6(qm + x) + 1 \equiv 6(jm + x) + 1 \pmod{n}$$

since otherwise we would arrive at a contradiction $q \equiv j \pmod{n}$. Thus the terms of the sequence, x, $m + x$, $2m + x$, . . . , $(n - 1)m + x$ are congruent modulo n to 0, 1, 2, . . . $n - 1$ in some order.

Now suppose t is congruent modulo n to $qm + x$. Then the integers, $6(qm + x) - 1$ and $6(qm + x) + 1$ are both relatively prime to $6n$ if and only if both $6t - 1$ and $6t + 1$ are relatively prime to $6n$. The implication is that the x^{th} column contains as many pairs of integers that are relatively prime to $6n$ as does the set {(5, 7), (11, 13), . . . , $(6n - 1, 6n + 1)$}, namely $\phi_2(n)$ pairs. Thus the number of pairs of integers $(6x - 1, 6x + 1)$ in the matrix that are relatively prime to $6m$ and $6n$ is $\phi_2(m)\phi_2(n)$. This completes the proof of the theorem.

The following result immediately follows from Theorem 1.1 and Theorem 1.6:

Corollary 1.7: If the integer n > 1 has the prime factorization $n = 2^{k1} 3^{k2} p_3^{k3} \ldots p_r^{kr}$
then $\phi_2(n) = 2^{k1} 3^{k2} (p_3^{k3} - 2p_3^{k3-1}) \ldots (p_r^{kr} - 2p_r^{kr-1})$.

Conflict of Interests

The authors have not declared any conflict of interests.

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