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Viscosity Approximation Methods in Reflexive Banach Spaces

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Authors' contributions

Both authors contributed collaboratively to this work. They wrote in tandem with each other, the first draft of the manuscript and managed literature searches. The final manuscript was approved by both authors.

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Abstract

In this paper, we study viscosity approximation methods in reflexive Banach spaces. Let X be a reflexive Banach space which admits a weakly sequentially continuous duality mapping $j\colon X \to X^*$, C a nonempty closed convex subset of X, h_n, where $n \geq 1$ a sequence of contractions on C and T_n, $\quad n = 1, 2, 3, \cdots N$, for $N \in \mathbb{N}$, a finite family of commuting nonexpansive mappings on C. We show that under appropriate conditions on σ_n the explicit iterative sequence τ_n defined by

$$\tau_{n+1} = \sigma_n h_n(\tau_n) + (1 - \sigma_n) T_n \tau_n, \;\; n \geq 1, \;\; \tau_1 \in C$$

where $\sigma_n \in (0, 1)$ converges strongly to a common fixed point $\tau \in \bigcap_{k=1}^{N} F_{\tau_k}$. We consequently show that the results is true for an infinite family T_n, $\quad n = 1, 2, 3, \cdots$ of commuting nonexpansive mapping on C.

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1 Introduction

Let X with dual X^* denote a reflexive Banach space which admits a weakly sequentially continuous duality mapping $j\colon X \to X^*$, and let $J : X \to 2^{X^*}$ be the normalized duality mapping defined by $Jx = \{h \in X^* : \langle x, h\rangle = \|x\|\|h\|;\ \ \|x\| = \|h\|\},\ \ \forall\ \ x \in X$. The single-valued duality mapping will be denoted by j; and F_T will denote the set of fixed points of T given by $F_T = \{x \in X; Tx = x\}$. We denote the strong convergence of the sequence $\{x_n\}$ in X to $x \in X$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$.

Let X denote a real Banach and space C a closed convex nonempty subset of X. Given a mapping $T : C \longrightarrow C$, T is called φ-Lipschitzian if there exists a constant $\varphi > 0$ such that

$$\| Tx - Ty \| \leq \varphi \| x - y \|, \quad \forall \ \ pair \quad x, y \in X. \tag{1.1}$$

T is a nonexpansive if for every pair $x, y \in X$, $\| Tx - Ty \| \leq \| x - y \|$. T is said to be a contraction on C if a constant $\alpha \in (0, 1)$ exists such that for every pair $x, y \in X$, $\| Tx - Ty \| \leq \varphi \| x - y \|$.

We denote by $\prod_C$ the set of all contractions on C.

A. Moudafi [1] who pioneered the concept of viscosity approximation methods introduced the iterative procedure defined by

$$x_{n+1} = \frac{1}{1+\varepsilon_n} Tx_n + \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n),\ \ x_1 \in C$$

where ε_n is a sequence of positive numbers in $\mathbb{R}$ such that $\varepsilon_n \to 0$ as $n \to \infty$, f is a contraction on C and T is a nonexpansive mapping defined on C; and obtained results in Hilbert spaces.

In [2], H. K Xu proposed the iterative scheme $\{x_n\}$ given by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n,\ \ n \geq 1,\ \ x_1 \in C \tag{1.2}$$

where $f \in \prod_C$, α_n is a sequence in (0,1) and T is a nonexpansive mapping on C, and proved the strong convergence of x_n to a fixed point of T.

Again in [3], Chang proved that as $n \to \infty$

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T_n x_n,\ \ n \geq 1,\ \ x_1 \in C \tag{1.3}$$

converges to a common fixed point of the nonexpansive mappings T_n , $n = 1, 2, \cdots, N$ in a Banach space.

The purpose of this paper is to prove the strong convergence of the sequence

$$\tau_{n+1} = \sigma_n h_n(\tau_n) + (1 - \sigma_n)T_n\tau_n,\ \ n \geq 1,\ \ \tau_1 \in C \tag{1.4}$$

in a reflexive Banach space X which admits a weakly sequentially continuous duality mapping $j : X \to X^*$ to a common fixed point of the family T_n ,$n = 1, 2, \cdots, N$. This generalizes and improves several recent results. Particularly, it extends and improves Theorem 4.2 of [2] and Theorem 1 of [3].

2 Preliminaries

Lemma 2.1

(Xu, [2]) Let X be a Banach space and C a bounded closed convex subset of X. Let $T : C \to C$ be a nonexpansive mapping on C and $h \in \prod_C$. Given a real number $\kappa \in (0, 1)$ define a mapping $T_\kappa : C \to C$ by $T_\kappa x = \kappa h(x) + (1 - \kappa)Tx, \ \forall \ x \in C$. Then T_κ is a contraction on C.

Proof

Let $x, y \in C$. Then

$$\begin{aligned}\|T_\kappa x - T_\kappa y\| &= \|\kappa h(x) + (1-\kappa)Tx - [\kappa h(y) + (1-\kappa)Ty]\| \\ &= \|\kappa[h(x) - h(y)] + (1-\kappa)[Tx - Ty]\| \\ &\leq \kappa\alpha\|x - y\| + (1-\kappa)\|x - y\| \\ &= (\kappa\alpha + (1-\kappa))\|x - y\| \\ &= (1 - \kappa(1-\alpha))\|x - y\|\end{aligned}$$

Since $1 - \kappa(1 - \alpha) \in (0, 1)$, T_κ is a contraction. ■

Theorem 2.2

Let X be a reflexive Banach space with dual X^* and bidual X^{**}. Let C be a nonempty bounded closed convex subset of X. Let h_n be a sequence of contractions on C such that $h_n \in C^* \ \forall \ n \geq 1$ and $h_n(x) \leq h_{n+1}(x), \ \forall \ n \geq 1, \ x \in X$. If h_n converges pointwise on C to a contraction h then the convergence is uniform.

Proof

Let $f_n(x) = h(x) - h_n(x)$ for each $n \in \mathbb{N}$. Then f_n is a sequence of contractions on the compact set C and $f_n\ (x) \geq f_{n+1}(x) \geq 0$ for all $x \in X$ where $n \in \mathbb{N}$.
Moreover,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \{h(x) - h_n(x)\} = 0 \tag{2.1}$$

Let $M_n = \sup\{f_n(x) : x \in C\}$ and let $\varepsilon > 0$ be given.
Also let

$$E_n = \{x \in C : f_n(x) < \varepsilon\} = f_n^{-1}((-\infty, \varepsilon))$$

Then E_n is open for each n and $E_n \subset E_{n+1}$ since $f_n(x) \geq f_{(n+1)}(x)$.

Since for each $x \in C$, $\lim_{n\to\infty} f_n(x) = 0$ there exists $n \in \mathbb{N}$ such that $f_n(x) < \varepsilon$ which implies $x \in E_n$. Thus $\bigcup_{n=1}^{\infty} E_n$ is an open cover for C and $\bigcup_{n=1}^{\infty} E_n = C$. Since C is compact, it has a finite subcover and in view of the fact that $E_n \subset E_{n+1}$, the largest of these also covers C. Hence there is $N \in \mathbb{N}$ such that $E_N = C$ and this means that $f_N(x) < \varepsilon$ for all $x \in C$ and $n \geq N$. Thus $M_N \leq \varepsilon$ and since $M_n \geq 0$, $\lim_{n\to\infty} M_n(x) = 0$. This indicates that the sequence f_n converges uniformly to 0 on C and therefore the sequence of contractions h_n converges uniformly to h on C. ■

Lemma 2.3

(Chang, [3]) Let X be a Banach space and let C be a bounded closed convex subset of X. Let T_k, $k = 1, 2, \cdots, N$ be a finite family of commuting nonexpansive self mappings defined on C.
Then $\bigcap_{k=1}^{N} F_{T_\kappa} = F_T$ where $T = T_1T_2T_3 \cdots T_{N-1}T_N$

Proof

Let $x \in \bigcap_{k=1}^{N} F_{T_k}$. Then $x \in F_{T_\kappa}$, for $k = 1, 2, \cdots, N$.
Hence $T_kx = x$ for each $k = 1, 2, \cdots, N$. Thus $x \in F_T$ since $Tx = T_1T_2T_3 \cdots T_{N-1}T_Nx = x$ implying that $\bigcap_{k=1}^{N} F_{T_\kappa} \subset F_T$.
Next, suppose $x \in F_T$. Then $T_1T_2T_3 \cdots T_{N-1}T_Nx = x$.
Letting $T = T_1$ gives $Tx = T_1x = x$. Thus $x \in F_{T_1}$
Again letting $T = T_1T_2$ implies that $x = Tx = T_1T_2x = T_2T_1x = T_2x$. Hence $x \in F_{T_2}$.
Next, if $T = T_1T_2T_3$ then we have $x = Tx = T_1T_2T_3x = T_3T_2T_1x = T_3T_2x = T_3x$ which implies $x \in F_{T_3}$.
Finally, for $N \in \mathbb{N}$ such that $N \geq 1$, let $T = T_1T_2T_3 \cdots T_{N-1}$ and assume that $T = T_1T_2T_3 \cdots T_{N-1}x = x$ implies $x \in F_{T_\kappa}$ for each $k = 1, 2, \cdots, N-1$.
Now, if $T = T_1T_2T_3 \cdots T_{N-1}T_N$ then

$$x = Tx = T_1T_2T_3 \cdots T_{N-1}T_Nx = T_NT_1T_2T_3 \cdots T_{N-1}x = T_Nx$$

Therefore $x \in F_{T_N}$. Thus, by induction if $T_1T_2T_3 \cdots T_{N-1}T_Nx = x$ then $T_kx = x$ for each $k = 1, 2, \cdots, N$ and so $x \in \bigcap_{k=1}^{N} F_{T_\kappa}$.
This follows that $F_T \subset \bigcap_{k=1}^{N} F_{T_\kappa}$ and the conclusion is that $\bigcap_{k=1}^{N} F_{T_\kappa} = F_T$. ■

Lemma 2.4

Let X be a Banach space and let C be a bounded closed convex subset of X. Let T_k, $k = 1, 2, \cdots, N$ be a finite family of commuting nonexpansive self mappings defined on C. Let $\bigcap_{k=1}^{N} F_{T_k} = F_T$, where $T = T_1T_2T_3 \cdots T_{N-1}T_N$. Then the mapping $T : C \to C$ is nonexpansive.

Proof

Let $x, y \in C$. Then $Tx = T_1T_2T_3 \cdots T_{N-1}T_Nx$, $Ty = T_1T_2T_3 \cdots T_{N-1}T_Ny$ and
$\|Tx - Ty\| = \|T_1T_2T_3 \cdots T_{N-1}T_Nx - T_1T_2T_3 \cdots T_{N-1}T_Ny\|$
Next we show that P_N is true $\forall\ N \in \mathbb{N}$.
Let $P_N : \|T_1T_2T_3 \cdots T_{N-1}T_Nx - T_1T_2T_3 \cdots T_{N-1}T_Ny\| \leq \|x - y\| \ \forall\ x, y \in C$
By hypothesis,

$$\|T_1x - T_1y\| \leq \|x - y\| \ \forall\ x, y \in C. \tag{2.2}$$

Therefore P_N is true when $N = 1$
Assuming that for some $k \in \mathbb{N}$, P_k is true we get

$$\|T_1T_2T_3 \cdots T_{k-1}T_kx - T_1T_2T_3 \cdots T_{k-1}T_ky\| \leq \|x - y\| \ \forall\ x, y \in C.$$

Finally by commutation of the nonexpansive maps;

$$\|T_1T_2T_3 \cdots T_kT_{k+1}x - T_1T_2T_3 \cdots T_kT_{k+1}y\| \leq \|T_{k+1}x - T_{k+1}y\| \leq \|x - y\| \ \forall\ x, y \in C$$

Thus $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$ proving that T is nonexpansive. ■

Lemma 2.5

(Xu, [2]) Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers, $\{\beta_n\}$ a sequence in (0,1) and $\{\varphi_n\}$ a sequence in $\mathbb{R}$ such that $\alpha_{n+1} \leq (1-\beta_n)\alpha_n + \varphi_n$, $n \geq 0$, where $\sum_{n=1}^{\infty} \beta_n = \infty$ and $\sum_{n=1}^{\infty} |\varphi_n| < \infty$. Then $\alpha_n \to 0$ as $n \to \infty$.

Lemma 2.6

(Yisheng, [4]) Let C be a nonempty closed convex subset of a reflexive Banach space X which satisfies Opial's condition. Suppose also that $T : C \to C$ is nonexpansive and $\{x_n\}$ is a sequence in C such that $x_n - Tx_n \to 0$. Then $x = Tx$

Proof

Let $x_n \rightharpoonup x$ and let $\lim_{n\to\infty} \sup \|x_n - Tx_n\| = 0$.
Now, since $\{x_n\}$ is bounded there exists $r > 0$ such that $\{x_n\} \subset \mathcal{A} := C \bigcap B_r$, where B_r is a closed ball of X with centre at 0 and radius r. Thus $\mathcal{A}$ is a nonempty bounded closed convex subset of C. For an arbitrary $\varepsilon > 0$, we choose $n_0 \in \mathbb{N}$ such that

$$\|x_n - Tx_n\| \leq \varepsilon, \ \ \forall \ \ n \geq n_0.$$

Let $\{\tau_n\}$ be a sequence in X such that $\tau_n \to \tau$ and $\|\tau_n - x_n\| < \frac{1}{n}$.
Again since $T : C \to C$ is nonexpansive and by extension Lipschitzian with Lipschitz constant φ,

$$\|Tx_n - T\tau_n\| \leq \varphi\|\tau_n - x_n\|.$$

Since

$$\|T\tau_n - \tau_n\| < \varepsilon \ \ \forall \ \ n, \ \ \lim_{n\to\infty} \sup \|T\tau_n - \tau_n\| < \varepsilon. \tag{2.3}$$

Finally,

$$\|Tx_n - x_n\| \leq \|Tx_n - T\tau_n\| + \|T\tau_n - \tau_n\| + \|\tau_n - x_n\| \leq \varphi\|x_n - \tau_n\| + \|T\tau_n - \tau_n\| + \|\tau_n - x_n\|$$

$$\implies \quad \|Tx_n - x_n\| \leq (1+\varphi)\|\tau_n - x_n\| + \|T\tau_n - \tau_n\|$$

By virtue of (2.3) and the fact that $\|\tau_n - x_n\| \to 0$ as $n \to \infty$, $\|Tx_n - x_n\| \leq \varepsilon$
Again, by virtue of the fact that $\lim_{n\to\infty} \sup \|Tx_n - x_n\| = 0 \; \forall \; n$,

$$\|Tx_n - x_n\| \leq \varepsilon \to 0 \ \ as \ \ n \to \infty \ \ and \ \ x = Tx.$$

■

Lemma 2.7

(Chang, [3]) Let X be a Banach space with dual X^*. Let $J : X \to 2^{X^*}$ defined by

$$J(x) = \{f \in X^* : (x, f) = \|x\|\|f\|, \|x\| = \|f\|\}, \ \ \forall \ x \in X$$

be the normalized duality mapping on X.

Then $\forall \ x \in X$, $\forall \ j(x) \in J(x)$ and $\forall \ j(x+y) \in J(x+y)$, the following subdifferetial inequalities (i) and (ii) hold in X.

(i) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle$

(ii) $\|x\|^2 + 2\langle y, j(x)\rangle \leq \|x+y\|^2$

Lemma 2.8

(Xu, [2]) Let X be a real Banach space with dual X^* and let J be as in Lemma (2.7). If C is a closed convex subset of X and $h : C \to C$ is a contraction with coefficient $\alpha \in (0, 1)$, then

$$(1-\alpha)\|x-y\|^2 \leq \langle (I-h)x-(I-h)y, j(x-y)\rangle \ \forall \ x, y \in C,$$

and $(I - h)$ is said to be strongly monotone where I is the identity operator.

3 Main Results

Theorem 3.1

Let X be a real reflexive Banach space which admits a weakly sequentially continuous duality mapping $j : X \to X^*$, C a bounded closed convex nonempty subset of X, h_n, $n \in \mathbb{N}$ a sequence of contractions on C such that $h_n\ (\tau) \leq h_{n+1}(\tau)$, $\forall\ n,\ \tau \in C$, and T_n, $n = 1, 2, 3, \cdots, N$ a finite family of commuting nonexpansive mappings on C such that $\bigcap_{n=1}^{N} F_{T_n} \neq \phi$ and satisfies the condition

$$\bigcap_{n=1}^{N} F_{T_n} = F(T_1T_2T_3\cdots T_{N-1}T_N) = F_T, \ \ where \ T = T_1T_2T_3\cdots T_{N-1}T_N.$$

Suppose also that for each $h_n \in \prod_C$, $h(p) \neq p$, $\forall\ p \in \bigcap_{n=1}^{N} F_{T_n}$ and σ_n is a real sequence in (0,1) that satisfies the following conditions;

$K_1 : \lim\limits_{n\to\infty} \sigma_n = 0$

$K_2 : \sum_{n=0}^{\infty} \sigma_n = \infty$

$K_3 : \|\sigma_{n+1} - \sigma_n\| < \infty$

Then the sequence τ_n defined by

$$\tau_{n+1} = \sigma_n h_n(\tau_n) + (1-\sigma_n)T_n\tau_n, \ \ \tau_1 \in C, \ \ n \geq 1 \tag{3.1}$$

satisfying the condition

$$K_4 : \ \ \|\tau_n - T\tau_n\| \to 0 \ \ as \ \ n \to \infty$$

converges strongly to $\tau^* \in \bigcap_{n=1}^{N} F_{T_n}$ such that τ^* is the unique solution of the variational inequality

$\langle (I-h)\tau^*, j(\tau^* - \tau)\rangle \leq 0$, $\forall\ \tau \in \bigcap_{n=1}^{N} F_{T_n}$.

Proof

From Theorem 2.2, $h_n \to h$ uniformly on C. Thus (3.1) becomes

$$\tau_{n+1} = \sigma_n h(\tau_n) + (1-\sigma_n)T_n\tau_n \tag{3.2}$$

To show that $\{\tau_{n+1}\}$ is bounded we choose $p \in \bigcap_{n=1}^{N} F_{T_n}$ to compute $\|\tau_{n+1} - p\|$.
It follows that

$$\begin{aligned}
\|\tau_{n+1} - p\| &= \|\sigma_n h(\tau_n) + (1-\sigma_n)T_n\tau_n - p\| \\
&= \|\sigma_n(h(\tau_n) - p) + (1-\sigma_n)(T_n\tau_n - p)\| \\
&\leq \sigma_n\|h(\tau_n) - p\| + (1-\sigma_n)\|T_n\tau_n - p\| \\
&\leq \sigma_n(\|h(\tau_n) - h(p) + h(p) - p\|) + (1-\sigma_n)\|\tau_n - p\| \\
&\leq \sigma_n\|h(\tau_n) - h(p)\| + \sigma_n\|h(p) - p\| + (1-\sigma_n)\|\tau_n - p\|
\end{aligned}$$

$$= \sigma_n\alpha\|\tau_n - p\| + \sigma_n\|h(p) - p\| + (1-\sigma_n)\|\tau_n - p\|$$
$$= (1-(1-\alpha)\alpha_n)\|\tau_n - p\| + \sigma_n\|h(p) - p\|$$
$$\leq \max\{\|\tau_n - p\|, \frac{1}{1-\alpha}\|h(p) - p\|\}$$

Thus by induction,

$$\|\tau_n - p| \leq \max\{\|\tau_1 - p\|, \frac{1}{1-\alpha}\|h(p) - p\|\}, n \geq 1.$$

Therefore $\{\tau_n\}$ is bounded and so are $\{T\tau_n\}$ and $\{h(\tau_n)\}$ where $T = T_1T_2T_3\cdots T_{N-1}T_N$.
Now,

$$\|\tau_{n+1} - \tau_n\| = \|\sigma_n h(\tau_n) + (1-\sigma_n)T\tau_n - \sigma_{n-1}h(\tau_{n-1}) - (1-\sigma_{n-1})T\tau_{n-1}\|$$
$$= \|(1-\sigma_n)(T\tau_n - T\tau_{n-1}) + (\sigma_n - \sigma_{n-1})(h(\tau_{n-1}) - T\tau_{n-1}) + \sigma_n(h(\tau_n) - h(\tau_{n-1}))\|$$
$$\leq (1-\sigma_n)\|\tau_n - \tau_{n-1}\| + K|\sigma_n - \sigma_{n-1}| + \sigma_n\alpha\|\tau_n - \tau_{n-1}\|,$$
$$= (1-(1-\alpha)\alpha_n)\|\tau_n - \tau_{n-1}\| + K|\sigma_n - \sigma_{n-1}| \qquad ;$$

where $K = \sup_{n\geq 1}\|h(\tau_{n-1}) - T\tau_{n-1}\| > 0$.
Thus

$$\|\tau_{n+1} - \tau_n\| = (1-(1-\alpha)\alpha_n)\|\tau_n - \tau_{n-1}\| + K|\sigma_n - \sigma_{n-1}|$$

Lemma 2.5 together with conditions K_2 and K_3 ensures that $\|\tau_{n+1} - \tau_n\| \to 0$.
Again,

$$\|\tau_n - T\tau_n\| = \|\tau_n - \tau_{n+1} + \tau_{n+1} - T\tau_n\| \leq \|\tau_n - \tau_{n+1}\| + \|\tau_{n+1} - T\tau_n\|$$
$$= \|\tau_n - \tau_{n+1}\| + \|\sigma_n h(\tau_n) + (1-\sigma_n)T\tau_n - T\tau_n\|$$
$$= \|\tau_n - \tau_{n+1}\| + \sigma_n\|h(\tau_n) - T\tau_n\| \to 0 \ \ as \ \ n \to \infty$$

Thus, $\|\tau_n - T\tau_n\| \to 0 \ \ as \ \ n \to \infty$ and condition K_4 is proved.
Next we show that

$$\lim_{n\to\infty}\sup\langle\tau^* - h(\tau^*), j(\tau_n - \tau^*)\rangle \geq 0, \ \ \forall \ \ \tau^* \in F_T = \bigcap_{n=1}^{N} F_{T_n}.$$

To do this we take a subsequence $\{\tau_{n_k}\}$ of $\{\tau_n\}$ such that

$$\lim_{n\to\infty}\sup\ \langle\tau^* - h(\tau^*), j(\tau_n - \tau^*)\rangle = \lim_{k\to\infty}\sup\ \langle\tau^* - h(\tau^*), j(\tau_{n_k} - \tau^*)\rangle$$

Now, since X is reflexive and τ_n is bounded, we may assume that $\tau_{n_\kappa} \rightharpoonup \tau^* as \ \ k \to \infty$.
Therefore by K_4 and Lemma 2.6, $\tau^* \in F_T = \bigcap_{n=1}^{N} F_{T_n}$.

By the fact that the duality mapping j is weakly sequentially continuous from X to X^*, we obtain

$$\lim_{n\to\infty}\sup\ \langle\tau^* - h(\tau^*), j(\tau_n - \tau^*)\rangle = \langle\tau^* - h(\tau^*), j(\tau^* - \tau^*)\rangle \geq 0. \tag{3.3}$$

Finally, $\tau_n \to \tau^*$ is shown below.
Now,

$$\|\tau_{n+1} - \tau^*\|^2 = \|\sigma_n h(\tau_n) + (1-\sigma_n)T_n\tau_n - \tau^*\|^2 = \|(1-\sigma_n)(T_n\tau_n - \tau^*) + \sigma_n(h(\tau_n) - \tau^*)\|^2$$
$$\leq (1-\sigma_n)^2\|(T_n\tau_n - \tau^*)\|^2 + 2\sigma_n\langle h(\tau_n) - \tau^*, j(\tau_{n+1} - \tau^*)\rangle$$
$$\leq (1-\sigma_n)^2\|\tau_n - \tau^*\|^2 + 2\sigma_n\langle h(\tau_n) - h(\tau^*) + h(\tau^*) - \tau^*, j(\tau_{n+1} - \tau^*)\rangle$$
$$\leq (1-\sigma_n)^2\|\tau_n - \tau^*\|^2 + 2\sigma_n\langle h(\tau_n) - h(\tau^*), j(\tau_{n+1} - \tau^*)\rangle + 2\sigma_n\langle h(\tau^*) - \tau^*, j(\tau_{n+1} - \tau^*)\rangle$$
$$\leq (1-\sigma_n)^2\|\tau_n - \tau^*\|^2 + 2\sigma_n\alpha\|\tau_n - \tau^*\|\|\tau_{n+1} - \tau^*\| + 2\sigma_n\langle h(\tau^*) - \tau^*, j(\tau_{n+1} - \tau^*)\rangle$$

Thus,

$\|\tau_{n+1}-\tau^*\|^2 \leq (1-\sigma_n)^2\|\tau_n-\tau^*\|^2+\alpha\sigma_n\{\|\tau_n-\tau^*\|^2+\|\tau_{n+1}-\tau^*\|^2\}+2\sigma_n\langle h(\tau^*)-\tau^*, j(\tau_{n+1}-\tau^*)\rangle$

By (Lemma 2.7 ii) we get,

$$(1-\alpha\sigma_n)\|\tau_{n+1}-\tau^*\|^2 \leq (1-2\sigma_n+\sigma_n^2)\|\tau_n-\tau^*\|^2+\alpha\sigma_n\|\tau_n-\tau^*\|^2+2\sigma_n\langle h(\tau^*)-\tau^*, j(\tau_{n+1}-\tau^*)\rangle \quad (3.4)$$

Since $j(-x) = -j(x)\ \forall\ x \in X$, (3.3) becomes

$$\lim_{n\to\infty} \sup\langle h(\tau^*) - \tau^*, j(\tau^* - \tau_n)\rangle \leq 0 \quad (3.5)$$

If

$$\varphi_n = \max\{\langle h(\tau^*) - \tau^*, j(\tau^* - \tau_n)\rangle, 0\} \geq 0,\ \ \forall\ \ n \geq 0,$$

then (3.5) ensures that for any given $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that

$$\langle h(\tau^*) - \tau^*, j(\tau^* - \tau_n)\rangle < \varepsilon,\ \ \forall\ \ n \geq N_1.$$

Therefore $0 \leq \varphi_n < \varepsilon\ \ \forall\ \ n \geq N_1$ and since ε is arbitrary $\varphi_n \to 0$ as $n \to \infty$.
Thus

$$\langle h(\tau^*) - \tau^*, j(\tau_{n+1} - \tau^*)\rangle \leq \varphi_n.$$

Therefore by (3.4),

$$(1 - \alpha\sigma_n)\|\tau_{n+1} - \tau^*\|^2 \leq (1 - 2\sigma_n + \sigma_n^2)\|\tau_n - \tau^*\|^2 + \alpha\sigma_n M + 2\sigma_n\varphi_n,$$

where $M = \sup_{n\geq 0} \|\tau_n - \tau^*\|^2$.

Which implies $\|\tau_{n+1} - \tau^*\|^2 \leq \dfrac{1 - 2\sigma_n + \sigma_n^2}{1 - \alpha\sigma_n}\|\tau_n - \tau^*\|^2 + \dfrac{1}{1 - \alpha\sigma_n}(\alpha\sigma_n M + 2\sigma_n\varphi_n)$.

Now, since $\alpha \in (0, 1)$ and $\sigma_n \to 0$ as $n \to \infty$ there exists $N_2 \in \mathbb{N}$ such that $1-\alpha\sigma_n > \dfrac{1}{2},\ \ \forall\ n \geq N_2$.
Hence $\dfrac{1}{1 - \alpha\sigma_n} < 2$ and $\dfrac{1}{1 - \alpha\sigma}(\alpha\sigma_n M + 2\sigma_n\varphi_n) < 2(\alpha\sigma_n M + 2\sigma_n\varphi_n)$.
Again, let

$$\frac{1 - 2\sigma_n + \sigma_n^2}{1 - \alpha\sigma_n} = \frac{1 - 2\sigma_n + \alpha\sigma_n}{1 - \alpha\sigma_n} = \frac{1 - \sigma_n(2 - \alpha)}{1 - \alpha\sigma_n} = 1 - \frac{2\alpha_n(1 - \alpha)}{1 - \alpha\sigma_n} \leq 1 - 2\alpha_n(1 - \alpha).$$

It follows from above analysis that

$$\|\tau_{n+1} - \tau^*\|^2 \leq \{1 - 2\alpha_n(1 - \alpha)\}\|\tau_n - \tau^*\|^2 + 2\sigma_n(\alpha M + 2\varphi_n),\ \forall\ n \geq N_2.$$

Thus, by Lemma 2.5, $\|\tau_n - \tau^*\| \to 0$ *as* $n \to \infty$ and consequently $\tau_n \to \tau^* \in \cap_{n=1}^{N} F_{T_n}$. ■

We next consider an infinite family $T_1, T_2, T_3, ...$ of nonexpansive commuting maps on C. For the purpose of achieving our objective, we group these mappings into equivalent classes each of size N where $N \in \mathbb{N}$. Thus the iteration of study becomes

$$\tau_{n+1} = \sigma_n h_n(\tau_n) + (1 - \sigma_n)T_n\tau_n,\ \ \tau_1 \in C, n \geq 1 \text{ where } T_n = T_n \pmod N.$$

Lemma 3.2

Let X be a nonempty set. Then all equivalent classes of X are disjoint and X is the union of its equivalent classes.

Proof

Let A be a nonempty subset of X. Then A is an equivalent class if there is an element $x \in A$ such that $\forall\ y \in A, x \sim y$. Thus, A consists precisely of elements which are equivalent to x.
Let A ,B be any two equivalence classes of X and suppose $A \cap B \neq \phi$. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. This implies $x \sim y\ \forall\ y \in A$ and $x \sim z\ \forall\ z \in B$. Therefore $y \sim x$ and $x \sim z$. Hence by transitivity $y \sim z$. This gives a contradiction since A and B are two distinct equivalent classes. Thus $A \cap B = \phi$. Finally, since $A \subset X$, $B \subset X$ and $A \cap B = \phi$ for every pair of subsets of X, X is the union of its equivalent classes.

Theorem 3.3

Let X be a real reflexive Banach space which admits a weakly sequentially continuous duality mapping $j : X \to X^*$, C a bounded closed convex nonempty subset of X, $h_n, n \in \mathbb{N}$, a sequence of contractions on C such that $h_n(\tau) \leq h_{n+1}(\tau),\ \forall\ n,\ \tau \in C$, and $T_n,\ n = 1, 2, 3, ...$, an infinite family of commuting nonexpansive self mappings on C such that $\cap_{n=1}^{N} F_{T_n} \neq \phi$ and satisfies the condition

$$\cap_{n=1}^{N} F_{T_n} = F(T_1T_2T_3...T_{N-1}T_N) = F_T,\ \text{where } T = T_1T_2T_3...T_{N-1}T_N.$$

Suppose also that for each $h_n \in \prod_C,\ h(p) \neq p,\ \forall\ p \in \cap_{n=1}^{N} F_{T_n}$ and σ_n is a real sequence in (0,1) that satisfies the following conditions;

$K_1 : \lim\limits_{n\to\infty} \sigma_n = 0.$

$K_2 : \sum_{n=0}^{\infty} \sigma_n = \infty$

$K_3 : \sum_{n=0}^{\infty} |\sigma_{n+1} - \sigma_n| < \infty$

Then the sequence

$$\tau_{n+1} = \sigma_n h_n(\tau_n) + (1 - \sigma_n)T_n\tau_n, \tau_1 \in C, n \geq 1,$$

satisfying the condition

$$K_4 : \|\tau_n - T\tau_n\| \to 0 \ \text{ as } \ n \to \infty$$

converges strongly to $\tau^* \in \cap_{n=1}^{N} F_{T_n}$ such that τ^* is the unique solution of the variational inequality

$$\langle (I - h)\tau^*, j(\tau^* - \tau)\rangle \leq 0, \forall \tau \in \cap_{n=1}^{N} F_{T_n}$$

Proof

The proof is done by partitioning the infinite family $T_n, n = 1, 2, 3, ...$ into equivalence classes of size N, for a positive integer N. Thus, all the assumptions made for the finite number of nonexpansive maps in Theorem 3.1 hold for each class. Again, since equivalent classes are disjoint, there are no spillovers into other classes. Therefore any result that is true for one class will also hold for other classes. Consequently, the result is apriori true from Theorem 3.1.

4 Conclusion

Halpern [5], Lions [6], Wittmann [7], Bauschke [8], Xu [2] and Chang [3] considered iterative sequences which were special cases of the sequence

$$\tau_{n+1} = \sigma_n h_n(\tau_n) + (1 - \sigma_n)T_n\tau_n,\ \tau_1 \in C\ n \geq 1 \qquad (4.1)$$

1. If X is a Hilbert space, $h_n = u \in C \; \forall \; n \geq 1$ and $N = 1$ then (4.1) is reduced to

$$\tau_{n+1} = \sigma_n u + (1 - \sigma_n)T\tau_n, \; \tau_1 \in C, \; n \geq 1 \tag{4.2}$$

which was studied by Halpern [5], Lions [6] and Wittmann [7].

2. In a Hilbert space X, if $h_n = u \in C \; \forall \; n \geq 1$ and T_n ,$n = 1, 2, 3, \cdots, N$ is a finite family of nonexpansive mappings on C with $\bigcap_{k=1}^{N} F_{T_k} \neq \emptyset$, then (4.1) is reduced to

$$\tau_{n+1} = \sigma_n u + (1 - \sigma_n)T_n\tau_n, \; \tau_1 \in C, \; n \geq 1 \tag{4.3}$$

which was considered by Bauschke [8].

3. If X is a uniformly smooth Banach space, C a nonempty closed convex subset of X, $T : C \to C$ is a nonexpansive mapping with $F_T \neq \emptyset$ and $h_n = h \in \prod_C \; \forall \; n \geq 1$, then (4.1) is reduced to

$$\tau_{n+1} = \sigma_n h(\tau_n) + (1 - \sigma_n)T\tau_n, \; \tau_1 \in C, \; n \geq 1 \tag{4.4}$$

which was studied by Xu [2].

4. Again, if X is a uniformly smooth Banach space, C a nonempty closed convex subset of X, $T : C \to C$ is a nonexpansive mapping with $F_T \neq \emptyset$ and $h_n = h \in \prod_C \; \forall \; n \geq 1,$, T_n , $n = 1, 2, 3, \cdots, N$ is

a finite family of nonexpansive self mappings on C with $\bigcap_{k=1}^{N} F_{T_k} \neq \emptyset$, then (4.1) is reduced to the special case

$$\tau_{n+1} = \sigma_n h(\tau_n) + (1 - \sigma_n)T_n\tau_n, \; \tau_1 \in C, \; n \geq 1 \tag{4.5}$$

which was studied by Chang [3].

In summary, the iterative sequence (4.1) is a more general sequence which contains (4.2), (4.3), (4.4) and (4.5) as special cases.

Competing Interests

Authors have declared that no competing interests exist.

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