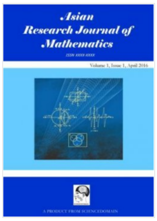

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A Generalization of Tychonoff's Product Theorem

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

A set-theoretical generalization of Tychonoff's theorem on compactness of the product of compact topological spaces is proved.

Keywords: Compact set; Tychonoff's product.

2010 Mathematics Subject Classification: 03E99, 54B10.

1 Introduction

The original proof of Tychonoff's product theorem relies on the Kuratowski – Zorn lemma, i. e, on the axiom of choice. Later it was shown (see [1] and the references therein) that this axiom can be eliminated from the proof (but not from the assertion). In [1], the notion of (pointless) topological space was generalized to this end. In the present article, we generalize it yet more (joins are not mentioned at all) and prove in this generality an elementary set-theoretical compactness theorem having no prototypes in the literature. As a topological application of this main result, we give a choice-free proof of the classical Tychonoff theorem.

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2 Preliminaries

A subset of all nonempty finite subsets of a set I will be denoted by $(2^I)_0$. The least element, provided it exists, of an ordered set will be denoted by $\mathbf{0}$, maybe with a subscript. The symbols inf and $\wedge$ are equipollent.

We will say that, by definition,

- an ordered set is *bottomed* if it contains the exact lower bound of every its subset;
- a subset (below referred to as a system) $\{x_i, i \in I\}$ of a bottomed lattice is *centered* if for any $J \in (2^I)_0 \quad \bigwedge\limits_{i\in J} x_i \neq \mathbf{0}$;
- a unary operation f in an ordered set X is: *inductive* (*idempotent*) if for any $x \in X \quad x \prec f(x)$ ($f(f(x)) = f(x)$, respectively), a *hypoclosing* if it is both inductive and idempotent;
- the element $f(x)$, where f is a hypoclosing, is a *hypoclosure* of x;
- an element coinciding with its hypoclosure is *hypoclosed*;
- an equipped with a inductive idempotent operation bottomed set is a *hypotopological space*;
- an element x of a hypotopological space is *hypocompact* if for any centered system Z of hypoclosed elements of this space
$$\bigwedge_{z\in Z} x \wedge z \neq \mathbf{0}; \tag{2.1}$$
- a set H of hypoclosed elements in a hypotopological space X is a *hypobase* of this space if for any $x \in X$ there exists $H_x \subset H$ such that $x = \bigwedge\limits_{y\in H_x} y$.

Lemma 2.1. *Let U be a bottomed set, V be its bottomed subset, and f be the operation in U defined by $f(u) = \inf\{v \in V : u \prec v\}$. Then f is idempotent.*

Proof. Obviously, $f(u) \prec f(f(u))$. For $x \prec f(f(u))$ and $w \in V$, the obvious implication $f(u) \prec w \;\Rightarrow\; x \prec w$ shows that $x \prec f(u)$. Thus $f(f(u)) \prec f(u)$. ☐

Proposition 2.1. *In order that an element x of a hypotopological space X be hypocompact it is necessary and sufficient that there exist a hypobase $\mathcal{F}$ in X such that for any centered system $Z \subset \mathcal{F}$ relation* (2.1) *holds.*

Proof. N e c e s s i t y follows from the obvious fact that every hypotopology is its own hypobase.

S u f f i c i e n c y. Let $\mathcal{F}$ be a hypobase. Then for any hypoclosed element y there exists $Z_y \subset \mathcal{F}$ such that $y = \bigwedge\limits_{z\in Z_y} z$. Let y range over some centered system Y. On the strength of the last equality $y \prec z$ as $z \in Z_y$, so the system $Z \equiv \bigcup\limits_{y\in Y} Z_y$ is also centered. So if for x and $\mathcal{F}$ are as in the assertion, then relation (2.1) holds. Its left-hand side is none other than $\bigwedge\limits_{y\in Y} x \wedge y$. ☐

Let X be a bottomed set, Θ be a nonempty set; for each $\theta \in \Theta$, X_θ be a hypotopological space with hypoclosing $\bar{\cdot}$ (denoted likewise for all θ), and $p_\theta : X \to X_\theta$ be an operator with the property
$$(\forall\, \theta \in \Theta \quad p_\theta y = p_\theta z) \;\Rightarrow\; y = z \tag{2.2}$$
(i. e., the family $\{p_\theta, \theta \in \Theta\}$ separates points of X).

Denote
$$X_* = \{x \in X : \exists\, y \in X \quad \forall\, \theta \in \Theta \quad p_\theta y = \overline{p_\theta x}\}$$

($\neq \emptyset$ since $\mathbf{0} \in X_*$). If for some $x \quad p_\theta y = \overline{p_\theta x}$ for all θ (the element y with this property is unique due to (2.2)), then we put by definition $[x]_* = y$. Otherwise speaking, for $x \in X_*$ the element $[x]_*$ is uniquely determined by the formula

$$p_\theta [x]_* = \overline{p_\theta x},\ \theta \in \Theta.$$

Denote further

$$H = \{x \in X_* : [x]_* = x\}, \quad \underline{X} = \left\{ \bigwedge_{y \in S} y,\ S \subset H \right\}$$

and define the operation $[\cdot]$ on X by

$$[x] = \inf\{z \in \underline{X} : x \prec z\} \equiv \bigwedge_{x \prec z \in \underline{X}} z. \tag{2.3}$$

Obviously, the set $\underline{X}$ is bottomed.

Lemma 2.2. *The operation* $[\cdot]$ *is a hypoclosing.*

Proof. Inductiveness is immediate from (2.3), idempotence – from (2.3) and Lemma 2.1. □

3 The Main Result

In this section, we use the Gothic font for the last three letters of the alphabet and for H. So

$$\mathfrak{X}_* = \{\mathfrak{x} \in \mathfrak{X} : \exists\, \mathfrak{y} \in \mathfrak{X} \quad \forall\, \theta \in \Theta \quad p_\theta \mathfrak{y} = \overline{p_\theta \mathfrak{x}}\},$$

$$\mathfrak{H} = \{\mathfrak{x} \in \mathfrak{X} : [\mathfrak{x}]_* = \mathfrak{x}\}, \quad \underline{\mathfrak{X}} = \left\{ \bigwedge_{\mathfrak{x} \in S} \mathfrak{x},\ S \subset \mathfrak{H} \right\};$$

for $\mathfrak{x} \in X_*$, the element (whose uniqueness will emerge from (3.1)) $[\mathfrak{x}]_*$ is defined by the formula

$$p_\theta [\mathfrak{x}]_* = \overline{p_\theta \mathfrak{x}},\ \theta \in \Theta;$$

the operation $[\cdot]$ is defined by

$$[x] = \inf\{\mathfrak{z} \in \underline{\mathfrak{X}} : \mathfrak{x} \prec \mathfrak{z}\} \equiv \inf\{\mathfrak{z} \in \mathfrak{H} : \mathfrak{x} \prec \mathfrak{z}\}.$$

Theorem 3.1. *Let the following objects be given: a nonempty set* Θ *and a bottomed set* $\mathfrak{X}$*; for each* $\theta \in \Theta$ *– a hypotopological space* $\mathfrak{X}_\theta$ *with a hypoclosing* $\bar{\cdot}$ *(denoted likewise for all* θ*) and an operator* $p_\theta : \mathfrak{X} \to \mathfrak{X}_\theta$*; a set* $\mathfrak{Z} \subset \mathfrak{X}$ *such that for any* $\theta \in \Theta \quad p_\theta \mathfrak{Z}$ *is a hypobase in* $\mathfrak{X}_\theta$*. Assume that the family* $\{p_\theta, \theta \in \Theta\}$ *has the properties:*

$$(\forall\, \theta \in \Theta \quad p_\theta \mathfrak{y} = p_\theta \mathfrak{z}) \;\Rightarrow\; \mathfrak{y} = \mathfrak{z}, \tag{3.1}$$

for any system $\{\mathfrak{y}_i, i \in I\} \subset \mathfrak{X}$ *there holds*

$$\left(\forall\, \theta \in \Theta \quad \bigwedge_{i \in I} p_\theta \mathfrak{y}_i \neq \mathbf{0}_\theta \right) \;\Rightarrow\; \bigwedge_{i \in I} \mathfrak{y}_i \neq \mathbf{0}. \tag{3.2}$$

Then: (i) The operation $[\cdot]$ *is a hypoclosing. (ii) If* $\mathfrak{x}$ *is an element of* $\mathfrak{X}$ *such that for each* $\theta \in \Theta \quad p_\theta \mathfrak{x}$ *is hypocompact, then* $\mathfrak{x}$ *is itself hypocompact.*

Note prior to the proof that property (3.1) provides correctness of the definition of $[\cdot]_*$ ($[\mathfrak{x}]_*$ is unique for any $\mathfrak{x} \in \mathfrak{X}_*$).

Proof. The first statement is the replica of Lemma 2.2, so we prove the second. Fix θ. By condition the element $p_\theta \mathfrak{x}$ is hypocompact. Let Z be a centered subsystem of $\mathfrak{Z}$. Then so is the system $p_\theta Z$ (since, for any mapping, the intersection of the images of arbitrary sets contains the image of their intersection), whence by the definition of hypocompactness

$$\bigwedge_{\mathfrak{z} \in Z} p_\theta \mathfrak{x} \wedge p_\theta \mathfrak{z} \neq \mathbf{0}_\theta,$$

which together with (3.2) and arbitrariness of θ yields

$$\bigwedge_{\mathfrak{z} \in Z} \mathfrak{x} \wedge \mathfrak{z} \neq \mathbf{0}.$$

It remains to refer to Proposition 2.1. □

Note that the proof of the theorem does not rely on the axiom of choice.

4 Tychonoff's Theorem as a Consequence of Theorem 3.1

Let T be an infinite set, and $\{X_t, t \in T\}$ be a family of nonempty sets. A point of the Cartesian product $\prod\limits_{t \in T} X_t$ will be denoted by $x(\cdot)$. It is a mapping of T into $\bigcup\limits_{t \in T} X_t$ such that for each $t \in T \quad x(t) \in X_t$. Take $\left(2^T\right)_0$ as Θ, the Boolean of $\prod\limits_{t \in T} X_t$ as $\mathfrak{X}$, the Boolean of $\prod\limits_{t \in \theta} X_t$ as $\mathfrak{X}_\theta$. The sets $\mathfrak{X}$ and $\mathfrak{X}_\theta, \theta \in \Theta$, are ordered with $\subset$ as $\prec$. Under the above choice of $\mathfrak{X}$, its elements denoted in Section 3 by the lowercase Gothic letters turn out to be subsets of $\prod\limits_{t \in T} X_t$, so we use for them the customary symbols A, B etc. Define, for each $\theta \in \Theta$ and $A \subset \mathfrak{X}$, $p_\theta A$ by

$$p_\theta A = \{x(\cdot)|_\theta,\ x(\cdot) \in A\},$$

so that $\mathfrak{X}_\theta = p_\theta \mathfrak{X}$. Condition (3.1) takes the form

$$\left(\forall\, \theta \in \left(2^T\right)_0 \quad \{x(\cdot)|_\theta, x(\cdot) \in A\} = \{x(\cdot)|_\theta, x(\cdot) \in B\}\right) \Rightarrow A = B.$$

Let us prove even the stronger relation

$$\left(\forall\, \theta \in \left(2^T\right)_0 \quad \{x(\cdot)|_\theta, x(\cdot) \in A\} \subset \{x(\cdot)|_\theta, x(\cdot) \in B\}\right) \Rightarrow A \subset B.$$

If the antecedent is true, then one can write the obvious implications

$$x(\cdot) \notin B \Rightarrow \exists\, s \in T\ x(s) \notin p_{\{s\}} B \Rightarrow \exists\, \theta \in \left(2^T\right)_0\ x(\cdot)|_\theta \notin p_\theta A \Rightarrow x(\cdot) \notin A.$$

Condition (3.2) takes the form

$$\left(\forall\, \theta \in \left(2^T\right)_0 \quad \bigcap_{i \in I} p_\theta B_i \neq \emptyset\right) \Rightarrow \bigcap_{i \in I} B_i \neq \emptyset.$$

Here is the proof of this relation. Let us take, for each $s \in T$, a point $x_s \in \bigcap\limits_{i\in I} p_{\{s\}}B_i$ and define $x(\cdot)$ by $x(s) = x_s$. By construction $x(\cdot)|_{\{s\}} = x_s$, so for all $s \in T$ $x(\cdot)|_{\{s\}} \in p_{\{s\}}B_i$, $i \in I$, and therefore $x(\cdot) \in B_i$, $i \in I$.

From this time on, the X_t's are topological spaces, and $\bar{\cdot}$ is the topological closing in every $\prod\limits_{t\in\theta} X_t$, $\theta \in \Theta$, as well as in their Tychonoff's product. For $E \in X_t$, E^c and $C_t(E)$ signify $X_t \setminus E$ and $\left\{x(\cdot) \in \prod\limits_{t\in T} X_t : x(t) \in E\right\}$, respectively. For a subset E either of X_t or of $\prod\limits_{t\in T} X_t$, E° means the interior of E.

Lemma 4.1. *Let, for each $t \in T$, F_t be a closed set in a topological space X_t. Then $\prod\limits_{t\in T} F_t$ is a closed set in $\prod\limits_{t\in T} X_t$ equipped with Tychonoff's topology.*

Proof. $\prod X_t \setminus \prod F_t = \bigcup C_t(F_t^c)$. ☐

The next statement is obvious.

Lemma 4.2. *For any $t \in T$ and $B_t \in X_t$ the set $C_t(B_t)$ is open iff so is B_t.*

Corollary 4.3. *For any $t \in T$ and $B_t \in X_t$ $(C_t(B_t))^\circ \subset C_t(B_t^\circ)$.*

Proof. The left-hand side is the union of all open subsets of $C_t(B_t)$, and the collection of the latter is, by Lemma 4.2, exhausted by sets of the kind $C_t(G_t)$, where G_t is an open subset of B_t. ☐

Lemma 4.4. *Let, for each $t \in T$, A_t be a set in a topological space X_t. Then $\prod\limits_{t\in T} \overline{A_t} = \overline{\prod\limits_{t\in T} A_t}$.*

Proof. Denote $Q = \prod\limits_{t\in T} A_t$, $R = \prod\limits_{t\in T} \overline{A_t}$. In this proof, the superscript c will be written after the symbols of subsets of $\prod X_t$, as well, and will signify the complement to $\prod X_t$. One has $\overline{Q} \subset R$ since $Q \subset R$ and R is closed by Lemma 4.1. On the other hand, $R^c = \bigcup C_t\left(\overline{A_t}^c\right) = \bigcup C_t(A_t^{c\circ})$,

$$\overline{Q}^c = Q^{c\circ} = (C_t(A_t^c))^\circ \supset \bigcup (C_t(A_t^c))^\circ.$$

It remains to apply Corollary 4.3 to $B_t = A_t^c$. ☐

Denote

$$\mathfrak{X}_* = \left\{\bigcup_{k=1}^m \prod_{t\in T} D_{kt} : m \in \mathbb{N},\ D_{kt} \subset X_t\right\}.$$

Obviously,

$$p_\theta \bigcup_{k=1}^m \prod_{t\in T} D_{kt} = \bigcup_{k=1}^m \prod_{t\in\theta} D_{kt}.$$

Hence we get by Lemma 4.4

$$\overline{p_\theta \bigcup_{k=1}^m \prod_{t\in T} D_{kt}} = \bigcup_{k=1}^m \prod_{t\in\theta} \overline{D_{kt}} \equiv p_\theta \bigcup_{k=1}^m \prod_{t\in T} \overline{D_{kt}}.$$

So, for

$$\mathfrak{x} = \bigcup_{k=1}^m \prod_{t\in T} D_{kt}$$

one has

$$[\mathfrak{x}]_* = \bigcup_{k=1}^{m} \prod_{t\in T} \overline{D_{kt}}.$$

Put

$$\mathfrak{Z} = \left\{ \bigcup_{k=1}^{m} \prod_{t\in T} F_{kt} : m \in \mathbb{N},\, F_{kt} \subset X_t,\ \text{the } F_{kt}\text{'s are closed} \right\}.$$

Let us show that for each $\theta \in \Theta$ $\;p_\theta \mathfrak{Z}$ is a hypobase in $\mathfrak{X}_\theta$. To this end we will show even more: $\{Q^c,\, Q \in p_\theta \mathfrak{Z}\}$ is a base in $\mathfrak{X}_\theta$. Obviously,

$$p_\theta \mathfrak{Z} = \left\{ \bigcup_{k=1}^{m} \prod_{t\in \theta} F_{kt} : m \in \mathbb{N},\, F_{kt} \subset X_t,\ \text{the } F_{kt}\text{'s are closed} \right\},$$

$$\prod_{t\in\theta} X_t \setminus \bigcup_{k=1}^{m} \prod_{t\in\theta} F_{kt} = \bigcap_{k=1}^{m} \bigcup_{t\in\theta} p_\theta C_t(F_{kt}^c).$$

The set F_{kt}^c being open, so is the set on the right-hand side. Writing, for arbitrary $R_{kt} \subset X_t$ ($k \in \{1, \ldots, m\}$, $t \in \theta$), the identity

$$\bigcap_{k=1}^{m} \bigcup_{t\in\theta} R_{kt} = \bigcup_{f(\cdot)\in\theta^m} \bigcap_{k=1}^{m} R_{kf(k)},$$

we see that the sets $\bigcap\limits_{k=1}^{m} \bigcup\limits_{t\in\theta} p_\theta C_t(G_{kt})$ create a base of $\mathfrak{X}_\theta$ since so do even the sets $\bigcap\limits_{k=1}^{m} p_\theta C_{t_k}(G_{kt_k})$. Obviously, the hypotopological space $\mathfrak{X}$ with the above hypobase $\mathfrak{Z}$ is the same as the topological space $\mathfrak{X}$ with the Tychonoff's topology. Herein, for a topological space with closed sets in the role of hypoclosed ones, hypocompactness is tantamount to compactness. Thus we have deduced from Theorem 3.1 the following intermediate statement on the way to Tychonoff's theorem.

Lemma 4.5. *Let X_t, $t \in T$, be compact topological spaces such that for any $\theta \in \left(2^T\right)_0$ $\;\prod\limits_{t\in\theta} X_t$ is compact. Then $\prod\limits_{t\in T} X_t$ is compact in the Tychonoff topology.*

Applying this statement to $T = \{1, 2\}$, we get

Corollary 4.6. *The Tychonoff product of two compact topological spaces is compact.*

Hence we deduce by induction

Corollary 4.7. *The Tychonoff product of a finite number of compact topological spaces is compact.*

Juxtaposing Lemma 4.4 with Corollary 4.6, we arrive at the theorem of Tychonoff [2, 3]: *The Tychonoff product of compact topological spaces is itself compact.* Note that the deduction does not rely on the axiom of choice. In fact, this axiom underlies only the assertion of the theorem in its general form where it postulates nonemptyness of the Cartesian product. But in the cases when all the X_t's coincide or all of them are closed intervals the axiom is unnecessary even in the assertion. In the proof of the Banach – Alaoglu theorem [3, Th. 3.15], just the second situation occurs. So that theorem turns out independent of the axiom of choice.

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Competing Interests

Author has declared that no competing interests exist.

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