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An Exponential Attractor for a Two-Temperature Phase Transition Model

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

In this work, we investigate the finite dimensionality of an attractor of a two-temperature Caginalp-type system for heat conduction. In order to prove that the global attractor is of finite dimension, we can use the volume contraction method, show the existence of an inertial manifold or an exponential attractor. The

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volume contraction method is not applicable because it requires a certain differentiability of the associated semigroup, which is not possible to obtain for our system. Similarly, the construction of an inertial manifold relies on the so-called spectral gap condition, which is a very restrictive condition. For all these reasons, we show that the global attractor of the system is of finite fractal dimension by proving that the system has an exponential attractor.

Keywords: Transition model; two temperatures; global attractor; exponential attractor; Hausdorff's dimension; fractal dimension.

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1 Introduction

The asymptotic behaviour of dynamical systems is an important aspect of their study. This is why researchers often seek to demonstrate the existence of an attractor. An attractor is a compact region in the space of physical variables that attracts all or some of the system's trajectories. Determining its dimension is a key step in describing it. Since the attractor is a fractal object, its dimension is not integer.

Consider the Caginalp field-phase system defined by :

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = \varphi - \Delta\varphi \text{ in } \Omega, \tag{1.1}$$

$$\frac{\partial \varphi}{\partial t} - \Delta\frac{\partial \varphi}{\partial t} - \Delta\varphi = -\frac{\partial u}{\partial t} \text{ in } \Omega, \tag{1.2}$$

$$u = \varphi = 0 \text{ on } \partial\Omega, \tag{1.3}$$

$$u|_{t=0} = u_0,\ \varphi|_{t=0} = \varphi_0 \text{ in } \Omega. \tag{1.4}$$

The derivation of this system, the well-posdness of the problem and the asympotic behaviour have been done in [1]. In particular, we have shown the existence of a global attractor. Beyond the existence of the attractor, it is often interesting to estimate its dimension. Indeed, the dimension of a global attractor is an interesting geometrical property insofar as it gives information on the number of degrees of freedom defining the considered dynamic system (see [2], [3], [4], [5], [6]). The dimension of an attractor is understood to be an overlapping dimension such as the Hausdorff dimension or the fractal dimension (see [7], [8], [9], [10]). There are many techniques for estimating the dimension of a global attractor (see [11], [12], [13], [14]). One of them is to show the existence of an exponential attractor. Indeed, the existence of an exponential attractor implies not only the existence of a global attractor but also that the latter is of finite dimension (see [15], [4], [16], [17], [18], [19]).

In the first section of this work, we recall the existence result of the global attractor obtained in [1]. Finally, in the last section, we establish the existence of an exponential attractor, which allows us to conclude that the global attractor is of finite fractal dimension.

The letter c, or c', denotes a constant which may change from one line to another. Similarly, $\|.\|_p$ will denote the L^p norm and $(.,.)$ the usual scalar product L^2. We will denote by $\|.\|_X$ the norm in the Banach space X. Finally, when there is no possible confusion, we will note $\|.\|$ instead of $\|.\|_2$.

2 Preliminaries

In this section, we recall a certain number of assumptions and results obtained in [1]. They are an essential prerequisite for the rest of our work. These assumptions are as follows

$$-c_0 \leq F(s) \leq f(s)s + c_1, \quad c_0, c_1 \geq 0,\ s \in \mathbb{R}, \text{ and } F(s) = \int_0^s f(\tau)\,d\tau, \tag{2.1}$$

$$|f'(s)| \leq c_2(|s|^{2p}+1), \quad c_2, p \geq 1, \quad s \in \mathbb{R}, \tag{2.2}$$

$$f' \geq -c_3, \quad c_3 \geq 0, \tag{2.3}$$

$$f(0) = 0. \tag{2.4}$$

Thanks to these, we have obtained the following results :

Theorem 2.1. *Let $T > 0$ be given. We assume that $u_0 \in H_0^1(\Omega)$, $\varphi_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $F(u_0) < +\infty$. Then, under assumptions (2.1)-(2.2), the problem (1.1)-(1.4) admits at least one solution denoted (u, φ) such as $u \in L^\infty(\mathbb{R}_+; H_0^1(\Omega))$, $\varphi \in L^\infty(\mathbb{R}_+; H^2(\Omega) \cap H_0^1(\Omega))$, $\frac{\partial u}{\partial t} \in L^2((0,T) \times \Omega)$ and $\frac{\partial \varphi}{\partial t} \in L^2(0, T; H_0^1(\Omega))$.*

Theorem 2.2. *We then have under conditions of Theorem 2.1 that the problem (1.1)-(1.4) possesses a unique solution (u, φ) with the above regularity.*

Remark 2.3. A consequence of the theorems 2.1 and 2.2 is that we can define a family of resolution operators

$$\begin{aligned} S(t) : &\Phi \longrightarrow \Phi \\ &(u_0, \varphi_0) \longmapsto (u(t), \varphi(t)), \quad t \geq 0, \end{aligned} \tag{2.5}$$

where $\Phi = H_0^1(\Omega) \times H^2(\Omega) \cap H_0^1(\Omega)$ and (u, φ) is the unique solution to the problem (1.1)-(1.4). Besides, this family of solving operators forms a contnuous semigroup i.e. $S(0) = Id$ and $S(t+\tau) = S(t) \circ S(\tau)$, $\forall\, t, \tau \geq 0$.

Theorem 2.4. *Under the assumptions of Theorems 2.1 and 2.2, the semigroup $S(t)$ is dissipative on Φ. In other words, the semigroup $S(t)$ has a bounded absorbing set $\mathcal{B}$ in Φ.*

Taking into account all the above, we can now state the existence theorem of the global attractor

Theorem 2.5. *Under conditions of theorem 2.2 and taking into account (2.2)-(2.4). Then the semigroup $S(t)$ defined onto $H_0^1(\Omega) \times H^2(\Omega)$ possesses the global attractor denoted $\mathcal{A}$ which is bounded in $H^2(\Omega) \times H^3(\Omega)$.*

Proof. For the proof of the theorems 2.1, 2.2, 2.4 and 2.5 see [1]. □

To have proved the existence of a global attractor is certainly interesting but it would be even better to be able to estimate its dimension. For that we will demonstrate the existence of an exponential attractor.

3 Estimation of the Global Attractor Dimension

The notion of exponential attractor was introduced by Eden et al (see [20]) with the aim of correcting certain defects of the global attractor, in particular its speed of attraction and its robustness. In practice, to evaluate the dimension of the global attractor, we use the volume contraction method. This consists of studying the evolution of infinitesimal volumes of dimension k in a neighbourhood of the global attractor: if the semigroup contracts volumes of dimension k, therefore its fractal dimension is less than k. This method generally gives the best estimates of dimension in terms of physical parameters (see [21]). Nevertheless, the volume contraction method requires a certain degree of differentiability in the associated semigroup, which is difficult, if not impossible, to achieve. The finite dimensionality of the global attractor can also be obtained by exibiting an inertial manifold or an exponential attractor. The inertial manifold is a smooth (at least Lipschitz) finite-dimensional manifold satisfying an asymptotic completeness property. However, all known constructions of inertial manifold are based on a very restrictive condition known as the spectral gap (see, for example, [22], [19]). The existence of exponential attractor requires weaker assumptions, namely some Lipschitz or Hölder property, which can be more easy to get. Historically, the construction of the exponential attractor was founded on the squeezying property, which basically says that either the higher modes are dominated by the lower modes, or the flow is exponentially contracted. It is non-constructible in the sense of Zorn's lemma and is only valid in Hilbert spaces, since it makes essential use of orthogonal projectors of finite rank. Another construction of an exponential attractor valid in Banach spaces has been proposed in [23], [24]. It consists in establishing a regularity property on the difference of two solutions which generalises the techniques proposed in [25]. In our case, in the absence of regularisation effects on the initial data linked to the highly damped term $-\Delta\frac{\partial\varphi}{\partial t}$, the methods mentioned above no longer work. For this reason, we will use a decomposition argument (see [11], [23], [26], [27], [28], [13]).

Definition 3.1. A compact set $\mathcal{M}$ is an exponential attractor for the semi-group $S(t)$ if :

1) It is of finite fractal dimension, i.e. $dim(\mathcal{M}) < +\infty$,
2) It is positively invariant, i.e. $S(t)\mathcal{M} \subset \mathcal{M}$, $t > 0$,
3) It exponentially attracts bounded subsets of the phase space Φ in the following sense: $\forall B \subset \Phi$, bounded $dist_\phi(S(t)B, \mathcal{M}) \leq Q(\|B\|_\phi)e^{-ct}$, $c > 0$ constant and Q monotone function independent of B, $dist_\phi$ is the Hausdorff semi-distance between the sets.

We start with stating result that will be useful in the sequel (see [8], [20], [18]).

Theorem 3.1. *Let V and H be two Banach spaces such that $S(t) : X \to X$ is a semigroup acting on a closed subset X of H. We assume :*

a) $S(t)u-S(t)v = \phi^1(t)+\phi^2(t)$, $\forall u, v \in X$, where $\|\phi^1(t)\|_X^2 \leq d(t)\|u-v\|_H^2$, d continuous function, $d(t) \to 0$ quand $t \to +\infty$ and $\|\phi^2(t)\|_X^2 \leq h(t)\|u-v\|_H^2$, h continuous function,

b) the application $(t, x) \mapsto S(t)x$ is lipschitz in space and Hölder in time on $[0, T] \times B$, $T > 0, \forall B \subset X$, bounded.

Then $S(t)$ has an exponential attractor $\mathcal{M}$ on X.

To prove the existence of exponential attractors in our case, we will rely on theorem 3.1. We have the following result

Theorem 3.2. *The semigroup $S(t)$, $t \geq 0$, corresponding to the problem (1.1)-(1.4) defined from X to itself satisfies a decomposition as in theorem 3.1, provided that $p \leq 1$ when $\Omega \subset \mathbb{R}^3$.*

Proof. Let $X = \cup_{t\geq t_0}B$, B being the bounded absorbing set. We have $X \subset \Phi$. We restrict $S(t) : X \to X$. Let (u_1, φ_1) and (u_2, φ_2) be two trajectories of the problem (1.1)-(1.3) with initial data (u_{01}, φ_{01}) and (u_{02}, φ_{02}). We pose $u = u_1 - u_2$, $\varphi = \varphi_1 - \varphi_2$, $u_0 = u_{01} - u_{02}$, $\varphi_0 = \varphi_{01} - \varphi_{02}$. Thus (u, φ) verifies :

$$\frac{\partial u}{\partial t} - \Delta u + f(u_1) - f(u_2) = \varphi - \Delta\varphi, \tag{3.1}$$

$$\frac{\partial \varphi}{\partial t} - \Delta\frac{\partial \varphi}{\partial t} - \Delta\varphi = -\frac{\partial u}{\partial t}, \tag{3.2}$$

$$u = \varphi = 0 \text{ on } \partial\Omega, \tag{3.3}$$

$$u|_{t=0} = u_0, \; \varphi|_{t=0} = \varphi_0. \tag{3.4}$$

Subsequently, we decompose (u, φ) as follows $(u, \varphi) = (v, \zeta) + (w, \rho)$ where (v, ζ) is a solution of :

$$\frac{\partial v}{\partial t} - \Delta v = \zeta - \Delta\zeta, \tag{3.5}$$

$$\frac{\partial \zeta}{\partial t} - \Delta\frac{\partial \zeta}{\partial t} - \Delta\zeta = -\frac{\partial v}{\partial t}, \tag{3.6}$$

$$v = \zeta = 0 \text{ on } \partial\Omega, \tag{3.7}$$

$$v|_{t=0} = u_0, \; \zeta|_{t=0} = \varphi_0. \tag{3.8}$$

and (w, ρ) satisfies :

$$\frac{\partial w}{\partial t} - \Delta w + f(u_1) - f(u_2) = \rho - \Delta\rho, \tag{3.9}$$

$$\frac{\partial \rho}{\partial t} - \Delta\frac{\partial \rho}{\partial t} - \Delta\rho = -\frac{\partial w}{\partial t}, \tag{3.10}$$

$$w = \rho = 0 \text{ on } \partial\Omega, \tag{3.11}$$

$$w(0) = \rho(0) = 0. \tag{3.12}$$

Multiply (3.5) and (3.6) by $\frac{\partial v}{\partial t}$ and $(\zeta - \Delta\zeta)$ respectively. Then by summing them and integrating over Ω, we obtain

$$\frac{d}{dt}\left(\|\nabla v\|^2 + \|\zeta\|^2 + 2\|\nabla\zeta\|^2 + \|\Delta\zeta\|^2\right) + 2\left(\|\frac{\partial v}{\partial t}\|^2 + \|\nabla\zeta\|^2 + \|\Delta\zeta\|^2\right) = 0. \tag{3.13}$$

Now, multiplying (3.5) by v, then integrating through Ω

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + 2\|\nabla v\|^2 = \int_\Omega (\zeta - \Delta\zeta)v\,dx. \tag{3.14}$$

From (3.14), we get the inequality

$$\frac{d}{dt}\|v\|^2 + \|\nabla v\|^2 \le c\|\nabla\zeta\|^2. \tag{3.15}$$

Multiply (3.15) by ε with $\varepsilon > 0$ small enough, then summing with (3.13), we arrive at the inequality

$$\frac{d}{dt}\left(\varepsilon\|v\|^2 + \|\nabla v\|^2 + \|\zeta\|^2 + 2\|\nabla\zeta\|^2 + \|\Delta\zeta\|^2\right) + \varepsilon\|\nabla v\|^2 + 2(\|\nabla\zeta\|^2 + \|\Delta\zeta\|^2) + 2\|\frac{\partial v}{\partial t}\|^2 \le c\|\nabla\zeta\|^2. \tag{3.16}$$

Let's put

$$E = \varepsilon\|v\|^2 + \|\nabla v\|^2 + \|\zeta\|^2 + 2\|\nabla\zeta\|^2 + \|\Delta\zeta\|^2. \tag{3.17}$$

By choosing ε such that $2 - \varepsilon c > 0$, we obtain

$$\frac{dE}{dt} + \varepsilon\|\nabla v\|^2 + (2 - \varepsilon c)\|\nabla\zeta\|^2 + 2\|\Delta\zeta\|^2 + 2\|\frac{\partial v}{\partial t}\|^2 \le 0. \tag{3.18}$$

Now, using Poincaré's inequality, we have

$$E = \varepsilon\|v\|^2 + \|\nabla v\|^2 + \|\zeta\|^2 + 2\|\nabla\zeta\|^2 + \|\Delta\zeta\|^2 \le c\left(\|\nabla v\|^2 + \|\nabla\zeta\|^2 + \|\Delta\zeta\|^2\right). \tag{3.19}$$

From (3.18)-(3.19), the following inequation can be derived

$$\frac{dE}{dt} + cE + 2\|\frac{\partial v}{\partial t}\|^2 \le 0. \tag{3.20}$$

And so in particular

$$\frac{dE}{dt} + cE \le 0. \tag{3.21}$$

Multiplying (3.21) by e^{ct}, we obtain

$$\frac{dE}{dt}e^{ct} + cEe^{ct} \le 0. \tag{3.22}$$

Let

$$E(t) + \le E(0)e^{-ct}. \tag{3.23}$$

However, we have

$$c(\|v(t)\|^2_{H^1} + \|\zeta(t)\|^2_{H^2}) \leq E(t) \leq c_1(\|v(t)\|^2_{H^1} + \|\zeta(t)\|^2_{H^2}). \tag{3.24}$$

Hence, using (3.23), we have

$$\|v(t)\|^2_{H^1} + \|\zeta(t)\|^2_{H^2} \leq d(t)(\|u_0\|^2_{H^1} + \|\varphi_0\|^2_{H^2}), \tag{3.25}$$

with $d(t) = e^{-ct}$.

In the same way, Multiplying (3.9) and (3.10) by $\frac{\partial w}{\partial t}$ and $(\rho - \Delta\rho)$ respectively, then summing and integrating through Ω, we get

$$\frac{1}{2}\frac{d}{dt}\left(\|w\|^2 + \|\rho\|^2 + \|\nabla\rho\|^2 + \|\Delta\rho\|^2\right) + \|\frac{\partial w}{\partial t}\|^2 + \|\nabla\rho\|^2 + \|\Delta\rho\|^2 = -\int_\Omega (f(u_1) - f(u_2))\frac{\partial w}{\partial t}\,dx. \tag{3.26}$$

However, we have

$$|\int_\Omega (f(u_1) - f(u_2))\frac{\partial w}{\partial t}\,dx| \leq \int_\Omega f'(\theta(u_1,u_2))|u_1 - u_2||\frac{\partial w}{\partial t}|\,dx. \tag{3.27}$$

Using (2.2) in (3.27), we get

$$|\int_\Omega (f(u_1) - f(u_2))\frac{\partial w}{\partial t}\,dx| \leq c\int_\Omega (|u_1|^{2p} + |u_2|^{2p} + 1)|u||\frac{\partial w}{\partial t}|\,dx. \tag{3.28}$$

The Hölder inequality is then used to establish that ($i = 1, 2,\ p = 1$)

$$\begin{aligned}\int_\Omega |u_i|^{2p}|u||\frac{\partial w}{\partial t}|\,dx &\leq \left(\int_\Omega |u_i|^{6p}\,dx\right)^{\frac{1}{3}}\left(\int_\Omega |u|^6\,dx\right)^{\frac{1}{6}}\left(\int_\Omega |\frac{\partial w}{\partial t}|^2\,dx\right)^{\frac{1}{2}}\\ &\leq \|u_i\|^{2p}_{L^{6p}}\|u\|_{L^6}\|\frac{\partial w}{\partial t}\|_{L^2}\\ &\leq \|u_i\|^{4p}_{H^1}\|u\|^2_{H^1} + \frac{1}{2}\|\frac{\partial w}{\partial t}\|^2.\end{aligned} \tag{3.29}$$

Taking into account (3.28)-(3.29), the equality (3.26) becomes

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|w\|^2 + \|\rho\|^2 + \|\nabla\rho\|^2 + \|\Delta\rho\|^2\right) + \|\frac{\partial w}{\partial t}\|^2 + \|\nabla\rho\|^2 + \|\Delta\rho\|^2\\ &\leq c(\|u_1\|^q_{H^1} + \|u_2\|^q_{H^1} + 1)\|u\|_{H^1},\ q \geq 1.\end{aligned} \tag{3.30}$$

This time, multiplying (3.9) by w and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 + \|\Delta w\|^2 + \int_\Omega (f(u_1) - f(u_2))w\,dx = \int_\Omega (\rho - \Delta\rho)w\,dx. \tag{3.31}$$

By similarity of the calculations made in (3.27)-(3.29), we obtain

$$|\int_\Omega (f(u_1) - f(u_2))w\,dx| \leq c(\|u_1\|^q_{H^1} + \|u_2\|^q_{H^1} + 1)\|u\|_{H^1},\ q \geq 1. \tag{3.32}$$

Also, using the Schwarz and Poincaré formulas

$$\int_\Omega (\rho - \Delta\rho)w\,dx = \int_\Omega \rho w\,dx + \int_\Omega \nabla\rho\nabla w\,dx \leq c\|\nabla\rho\|^2 + \frac{1}{2}\|\nabla w\|^2. \tag{3.33}$$

Using (3.32)-(3.33) in (3.31), we get

$$\frac{1}{2}\frac{d}{dt}\|w\|^2+\|\Delta w\|^2\leq c(\|u_1\|^q_{H^1}+\|u_2\|^q_{H^1}+1)\|u\|^2_{H^1}+c_1\|\nabla\rho\|^2. \tag{3.34}$$

Also, multiplying (3.9) by $-\Delta w$ and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla w\|^2+\|\Delta w\|^2-\int_\Omega(f(u_1)-f(u_2))\Delta w\,dx=-\int_\Omega(\rho-\Delta\rho)\Delta w\,dx. \tag{3.35}$$

By similarity of the calculations made in (3.27)-(3.29), we obtain

$$|\int_\Omega(f(u_1)-f(u_2))\Delta w\,dx|\leq c(\|u_1\|^q_{H^1}+\|u_2\|^q_{H^1}+1)\|u\|^2_{H^1}+\frac{1}{4}\|\Delta w\|^2, \tag{3.36}$$

$$|\int_\Omega(\rho-\Delta\rho)w\,dx|\leq c(\|\nabla\rho\|^2+\|\Delta\rho\|^2)+\frac{1}{4}\|\Delta w\|^2. \tag{3.37}$$

Taking into account (3.36)-(3.37) in (3.35), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla w\|^2+\|\Delta w\|^2\leq c(\|u_1\|^q_{H^1}+\|u_2\|^q_{H^1}+1)\|u\|^2_{H^1}+c_1(\|\nabla\rho\|^2+\|\Delta\rho\|^2). \tag{3.38}$$

Also, multiplying (3.9) by $-\Delta\frac{\partial w}{\partial t}$ and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\|\Delta w\|^2+\|\nabla\frac{\partial w}{\partial t}\|^2-\int_\Omega(f(u_1)-f(u_2))\Delta\frac{\partial w}{\partial t}\,dx=-\int_\Omega(\rho-\Delta\rho)\Delta\frac{\partial w}{\partial t}\,dx. \tag{3.39}$$

However, we have

$$-\int_\Omega(\rho-\Delta\rho)\Delta\frac{\partial w}{\partial t}\,dx=-\int_\Omega\rho\Delta\frac{\partial w}{\partial t}\,dx+\int_\Omega\Delta\rho\Delta\frac{\partial w}{\partial t}\,dx. \tag{3.40}$$

In addition

$$\begin{aligned}-\int_\Omega(f(u_1)-f(u_2))\Delta\frac{\partial w}{\partial t}\,dx&=\int_\Omega\nabla(f(u_1)-f(u_2))\nabla\frac{\partial w}{\partial t}\,dx\\&=\int_\Omega(f'(u_1)\nabla u_1-f'(u_2)\nabla u_2)\nabla\frac{\partial w}{\partial t}\,dx.\end{aligned} \tag{3.41}$$

However, we have

$$f'(u_1)\nabla u_1-f'(u_2)\nabla u_2=(f'(u_1)-f'(u_2))\nabla u_1+f'(u_2)\nabla u. \tag{3.42}$$

Taking into account (3.42), (3.1) then becomes

$$|\int_\Omega(f(u_1)-f(u_2))\Delta\frac{\partial w}{\partial t}\,dx|\leq c\int_\Omega(|u_1|^{2p}+|u_2|^{2p}+1)|u||\nabla u_1||\nabla\frac{\partial w}{\partial t}|\,dx. \tag{3.43}$$

Since $H^2(\Omega)$ is continuous embeddings into $L^\infty(\Omega)$ (Ω bounded set), (3.43) gives

$$|\int_\Omega(f(u_1)-f(u_2))\Delta\frac{\partial w}{\partial t}\,dx|\leq c(\|u_1\|^2_{H^2}+\|u_2\|^2_{H^2}+1)\|u\|_{H^1}\|u_1\|_{H^2}|+\|\nabla\frac{\partial w}{\partial t}|\||. \tag{3.44}$$

Now, by multiplying (3.10) by $\Delta^2\rho$ and integrating through Ω, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|\Delta\rho|^2\,dx+\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla\Delta\rho|^2\,dx+\int_\Omega|\nabla\Delta\rho|^2\,dx=\int_\Omega\nabla\frac{\partial w}{\partial t}\nabla\Delta\rho\,dx. \tag{3.45}$$

Hence

$$\frac{1}{2}\frac{d}{dt}(\|\Delta\rho\|^2+\|\nabla\Delta\rho\|^2)+\|\nabla\Delta\rho\|^2=\int_\Omega \nabla\frac{\partial w}{\partial t}\nabla\Delta\rho\,dx. \tag{3.46}$$

Similarly, by multiplying (3.10) by $-\Delta\frac{\partial w}{\partial t}$ and then integrating through Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta w\|^2+\|\nabla\frac{\partial w}{\partial t}\|^2-\int_\Omega (f(u_1)-f(u_2))\Delta\frac{\partial w}{\partial t}=\int_\Omega(\rho-\Delta\rho)\Delta\frac{\partial w}{\partial t}\,dx. \tag{3.47}$$

By summing (3.46) and (3.47), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\Delta w\|^2+\|\Delta\rho\|^2+\|\nabla\Delta\rho\|^2)+\|\nabla\frac{\partial w}{\partial t}\|^2+\|\nabla\Delta\rho\|^2\\&-\int_\Omega(f(u_1)-f(u_2))\Delta\frac{\partial w}{\partial t}=-\int_\Omega(\rho-\Delta\rho)\Delta\frac{\partial w}{\partial t}\,dx+\int_\Omega\nabla\frac{\partial w}{\partial t}\nabla\Delta\rho\,dx.\end{aligned} \tag{3.48}$$

Now,

$$\int_\Omega(f(u_1)-f(u_2))\Delta\frac{\partial w}{\partial t}=-\int_\Omega(\rho-\Delta\rho)\Delta\frac{\partial w}{\partial t}\,dx+\int_\Omega\nabla\frac{\partial w}{\partial t}\nabla\Delta\rho\,dx. \tag{3.49}$$

In the same way, multiplying (3.9) by $\frac{\partial w}{\partial t}$ and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla w\|^2+\|\frac{\partial w}{\partial t}\|^2+\int_\Omega(f(u_1)-f(u_2))\frac{\partial w}{\partial t}\,dx=\int_\Omega(\rho-\Delta\rho)\Delta\frac{\partial w}{\partial t}\,dx. \tag{3.50}$$

Multiplying (3.10) by $\rho-\Delta\rho$ and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}(\|\rho\|^2+2\|\nabla\rho\|^2+\|\Delta\rho\|^2)+\|\nabla\rho\|^2+\|\Delta\rho\|^2=-\int_\Omega(\rho-\Delta\rho)\frac{\partial w}{\partial t}\,dx. \tag{3.51}$$

Now, adding (3.50) and (3.51), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla w\|^2+\|\rho\|^2+2\|\nabla\rho\|^2+\|\Delta\rho\|^2)\\&+\|\frac{\partial w}{\partial t}\|^2+\|\nabla\rho\|^2+\|\Delta\rho\|^2=-\int_\Omega(f(u_1)-f(u_2))\frac{\partial w}{\partial t}\,dx.\end{aligned} \tag{3.52}$$

However, we have, owing to (2.2)

$$\begin{aligned}|\int_\Omega(f(u_1)-f(u_2))\frac{\partial w}{\partial t}\,dx|&\le\int_\Omega|f'(\theta(u_1,u_2))|\,|u_1-u_2|\,|\frac{\partial w}{\partial t}|\,dx\\&\le c\int_\Omega(|u_1|^{2p}+|u_2|^{2p}+1)|u|\,|\frac{\partial w}{\partial t}|\,dx.\end{aligned} \tag{3.53}$$

If $\Omega\subset\mathbb{R}^3$, we have using, Holder's, Young's inequalities, sobolev embedding and noticing that $p\le 1$, in that case (we will take $p=1$)

$$\begin{aligned}\int_\Omega|u_i|^{2p}|u|\,|\frac{\partial w}{\partial t}|\,dx&\le\left(\int_\Omega|u_i|^{6p}\,dx\right)^{\frac{1}{3}}\left(\int_\Omega|u|^6\,dx\right)^{\frac{1}{6}}\left(\int_\Omega|\frac{\partial w}{\partial t}|^2\,dx\right)^{\frac{1}{2}},\ i=1,2\\&\le\|u_i\|^{2p}_{L^{6p}(\Omega)}\|u\|_{L^6(\Omega)}\|\frac{\partial w}{\partial t}\|\\&\le c\|u_i\|^2_{H^1(\Omega)}\|u\|_{H^1(\Omega)}\|\frac{\partial w}{\partial t}\|,\ (H^1(\Omega)\subset L^6(\Omega))\\&\le c\|u_i\|^4_{H^1(\Omega)}\|u\|^2_{H^1(\Omega)}+\frac{1}{6}\|\frac{\partial w}{\partial t}\|^2.\end{aligned} \tag{3.54}$$

Considering (3.53) and (3.54), the equality (3.52) becomes

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla w\|^2+\|\rho-\Delta\rho\|^2)+\frac{1}{2}\|\frac{\partial w}{\partial t}\|^2+\|\nabla\rho\|^2+\|\Delta\rho\|^2\\&\le c(\|u_1\|^4_{H^1(\Omega)}+\|u_2\|^4_{H^1(\Omega)}+1)\|u\|^2_{H^1(\Omega)}.\end{aligned}\tag{3.55}$$

Multiplying (3.9) by w and by integrate over Ω

$$\frac{1}{2}\frac{d}{dt}\|w\|^2+\|\nabla w\|^2+\int_\Omega(f(u_1)-f(u_2))w\,dx=\int_\Omega\rho w\,dx+\int_\Omega\nabla\rho\nabla w\,dx.\tag{3.56}$$

Using Poincaré's, Cauchy-Schwartz's and Young's inequalities, we have

$$\begin{aligned}|\int_\Omega\rho w\,dx|&\le\int_\Omega|\rho||w|\,dx\\&\le c\int_\Omega|\nabla\rho||\nabla w|\,dx\\&\le c\|\nabla\rho\|^2+\frac{1}{6}\|\nabla w\|^2\end{aligned}\tag{3.57}$$

and

$$\begin{aligned}|\int_\Omega\nabla\rho\nabla w\,dx|&\le\int_\Omega|\nabla\rho||\nabla w|\,dx\\&\le c'\|\nabla\rho\|^2+\frac{1}{6}\|\nabla w\|^2.\end{aligned}\tag{3.58}$$

Collecting (3.56)-(3.58), we get

$$\frac{1}{2}\frac{d}{dt}\|w\|^2+\frac{2}{3}\|\nabla w\|^2+\int_\Omega(f(u_1)-f(u_2))w\,dx\le c\|\nabla\rho\|^2.\tag{3.59}$$

Now, If $\Omega\subset\mathbb{R}^3$, we take $p=1$ (actually, $p\le 1$, in that case), Holder's inequality yields

$$\begin{aligned}|-\int_\Omega(f(u_1)-f(u_2))w\,dx|&\le\int_\Omega|f(u_1)-f(u_2)|\,|w|\,dx\\&\le\int_\Omega|f'(\theta(u_1,u_2))|\,|u_1-u_2|\,|w|\,dx\\&\le c\int_\Omega(|u_1|^{2p}+|u_2|^{2p}+1)|u|\,|w|\,dx\\&\le c(\|u_1\|^2_{H^1(\Omega)}+\|u_2\|^2_{H^1(\Omega)}+1)\|u\|_{H^1(\Omega)}\|w\|.\end{aligned}\tag{3.60}$$

Applying Poincaré's and Young's inequalities, we write

$$\begin{aligned}|\int_\Omega(f(u_1)-f(u_2))w\,dx|&\le c(\|u_1\|^2_{H^1(\Omega)}+\|u_2\|^2_{H^1(\Omega)}+1)\|u\|_{H^1(\Omega)}\|w\|\\&\le c(\|u_1\|^2_{H^1(\Omega)}+\|u_2\|^2_{H^1(\Omega)}+1)\|u\|_{H^1(\Omega)}\|\nabla w\|\\&\le c(\|u_1\|^4_{H^1(\Omega)}+\|u_2\|^4_{H^1(\Omega)}+1)\|u\|^2_{H^1(\Omega)}+\frac{1}{6}\|\nabla w\|^2.\end{aligned}\tag{3.61}$$

Taking into account (3.61), estimate (3.59) becomes

$$\frac{1}{2}\frac{d}{dt}\|w\|^2+\frac{1}{2}\|\nabla w\|^2 \leq c(\|u_1\|^4_{H^1(\Omega)}+\|u_2\|^4_{H^1(\Omega)}+1)\|u\|^2_{H^1(\Omega)}+c'\|\nabla\rho\|^2. \tag{3.62}$$

Adding (3.55) and ϵ_1(3.62), where $\epsilon_1 > 0$ is small enough, we obtain

$$\frac{dE_1}{dt}+c(\|\nabla\rho\|^2+\|\Delta\rho\|^2)+\|\nabla w\|^2+\|\Delta\frac{\partial w}{\partial t}\|^2 \leq c'(\|u_1\|^4_{H^1(\Omega)}+\|u_2\|^4_{H^1(\Omega)}+1)\|u\|^2_{H^1(\Omega)}, \tag{3.63}$$

where $E_1=\epsilon_1\|w\|^2+\|\nabla w\|^2+\|\rho-\Delta\rho\|^2$, satisfies

$$c(\|w\|^2_{H^1(\Omega)}+\|\rho\|^2_{H^2(\Omega)}) \leq E_1 \leq c'(\|w\|^2_{H^1(\Omega)}+\|\rho\|^2_{H^2(\Omega)}). \tag{3.64}$$

Thus, (3.63) becomes

$$\frac{dE_1}{dt}+cE_1+\|\frac{\partial w}{\partial t}\|^2 \leq c'(\|u_1\|^4_{H^1(\Omega)}+\|u_2\|^4_{H^1(\Omega)}+1)\|u\|^2_{H^1(\Omega)}. \tag{3.65}$$

Now, multiplying (3.9) by $-\Delta w$ and integrating over Ω, one has

$$\frac{1}{2}\frac{d}{dt}\|\nabla w\|^2+\|\Delta w\|^2=\int_\Omega(f(u_1)-f(u_2))\Delta w\,dx\int_\Omega\nabla\rho\nabla w\,dx+\int_\Omega\Delta\rho\Delta w\,dx. \tag{3.66}$$

Thanks to (2.2), we have

$$|\int_\Omega(f(u_1)-f(u_2))\Delta w\,dx| \leq c\int_\Omega(|u_1|^{2p}+|u_2|^{2p}+1)|u|\,|\Delta w|\,dx. \tag{3.67}$$

If $\Omega\subset\mathbb{R}^3$, $p\leq 1$ in that case, we take $p=1$ and then have thanks to Holder's and Young's inequalities

$$\begin{aligned}\int_\Omega(|u_1|^{2p}+|u_2|^{2p}+1)|u|\,|\Delta w|\,dx &\leq c(\|u_1\|^2_{H^1(\Omega)}+\|u_2\|^2_{H^1(\Omega)}+1)\|u\|_{H^1(\Omega)}\|\Delta w\| \\ &\leq c(\|u_1\|^4_{H^1(\Omega)}+\|u_2\|^4_{H^1(\Omega)}+1)\|u\|^2_{H^1(\Omega)}+\frac{1}{4}\|\Delta w\|^2.\end{aligned} \tag{3.68}$$

Finally, we are led to an inequality of the type

$$\frac{1}{2}\frac{d}{dt}\|\nabla w\|^2+\|\Delta w\|^2 \leq c(\|u_1\|^4_{H^1(\Omega)}+\|u_2\|^4_{H^1(\Omega)}+1)\|u\|^2_{H^1(\Omega)}+c'(\|\nabla\rho\|^2+\|\Delta\rho\|^2). \tag{3.69}$$

Adding (3.65) and ϵ_2 (3.69), with $\epsilon_2 > 0$ is small enough. We find

$$\frac{dE_2}{dt}+cE_2+\|\Delta w\|^2+\|\frac{\partial w}{\partial t}\|^2 \leq c'(\|u_1\|^4_{H^1(\Omega)}+\|u_2\|^4_{H^1(\Omega)}+1)\|u\|^2_{H^1(\Omega)}. \tag{3.70}$$

where $E_2=E_1+\epsilon_2\|\nabla w\|^2$ satisfies an estimates similar to (3.64).

Multiplying (3.9) by $-\Delta\frac{\partial w}{\partial t}$ and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\|\Delta w\|^2+\|\nabla\frac{\partial w}{\partial t}\|^2=\int_\Omega(f(u_1)-f(u_2))\Delta\frac{\partial w}{\partial t}\,dx+\int_\Omega\nabla\rho\nabla\frac{\partial w}{\partial t}\,dx+\int_\Omega\Delta\rho\Delta\frac{\partial w}{\partial t}\,dx. \tag{3.71}$$

Multiplying (3.10) by $\Delta^2\rho$ and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|\Delta\rho\|^2 + \|\nabla\Delta\rho\|^2) + \|\nabla\Delta\rho\|^2 = -\int_\Omega \Delta\rho\Delta\frac{\partial w}{\partial t}\,dx. \tag{3.72}$$

Adding (3.71) and (3.72), we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\Delta w\|^2 + \|\Delta\rho\|^2 + \|\nabla\Delta\rho\|^2) + \|\nabla\Delta\rho\|^2 + \|\nabla\frac{\partial w}{\partial t}\|^2\\ &= \int_\Omega (f(u_1) - f(u_2))\Delta\frac{\partial w}{\partial t}\,dx + \int_\Omega \nabla\rho\nabla\frac{\partial w}{\partial t}\,dx.\end{aligned} \tag{3.73}$$

However,

$$\begin{aligned}\int_\Omega (f(u_1) - f(u_2))\Delta\frac{\partial w}{\partial t}\,dx &= -\int_\Omega \nabla(f(u_1) - f(u_2))\nabla\frac{\partial w}{\partial t}\,dx\\ &= -\int_\Omega (f'(u_1)\nabla u_1 - f'(u_2)\nabla u_2)\nabla\frac{\partial w}{\partial t}\,dx\end{aligned} \tag{3.74}$$

Furthermore,

$$f'(u_1)\nabla u_1 - f'(u_2)\nabla u_2 = (f'(u_1) - f'(u_2))\nabla u_1 + f'(u_2)\nabla u \tag{3.75}$$

and therefore

$$\begin{aligned}|\int_\Omega (f(u_1) - f(u_2))\Delta\frac{\partial w}{\partial t}\,dx| &\le \int_\Omega |f'(u_1) - f'(u_2)|\,|\nabla u_1|\,|\nabla\frac{\partial w}{\partial t}|\,dx\\ &+ \int_\Omega |f'(u_2)|\,|\nabla u|\,|\nabla\frac{\partial w}{\partial t}|\,dx.\end{aligned} \tag{3.76}$$

We then deduce, thanks to (2.2), Holder's and Young's inequalities that

$$\begin{aligned}&\int_\Omega |f'(u_1) - f'(u_2)|\,|\nabla u_1|\,|\nabla\frac{\partial w}{\partial t}|\,dx\\ &\le \int_\Omega |f'(\theta(u_1,u_2))|\,|u_1 - u_2|\,|\nabla u_1|\,|\nabla\frac{\partial w}{\partial t}|\,dx\\ &\le c(\|u_1\|^{2p}_{L^\infty(\Omega)} + \|u_2\|^{2p}_{L^\infty(\Omega)} + 1)\int_\Omega |u|\,|\nabla u_1|\,|\nabla\frac{\partial w}{\partial t}|\,dx\\ &\le c(\|u_1\|^4_{H^2(\Omega)} + \|u_2\|^4_{H^2(\Omega)} + 1)\|u\|^2_{H^1(\Omega)}\,\|u_1\|^2_{H^2(\Omega)} + \frac{1}{6}\|\nabla\frac{\partial w}{\partial t}\|^2.\end{aligned} \tag{3.77}$$

Analogously, we have

$$\begin{aligned}&\int_\Omega |f'(u_2)|\,|\nabla u|\,|\nabla\frac{\partial w}{\partial t}|\,dx\\ &\le \int_\Omega (|u_2|^{2p} + 1)\,|\nabla u|\,|\nabla\frac{\partial w}{\partial t}|\,dx\\ &\le c(\|u_2\|^{2p}_{L^\infty(\Omega)} + 1)\int_\Omega |\nabla u|\,|\nabla\frac{\partial w}{\partial t}|\,dx\\ &\le c(\|u_1\|^4_{H^2(\Omega)} + \|u_2\|^4_{H^2(\Omega)} + 1)\|u\|^2_{H^1(\Omega)} + \frac{1}{6}\|\nabla\frac{\partial w}{\partial t}\|^2\end{aligned} \tag{3.78}$$

and

$$\begin{aligned} |\int_\Omega \nabla\rho\nabla\frac{\partial w}{\partial t}\,dx| &\leq |\int_\Omega |\nabla\rho|\,|\nabla\frac{\partial w}{\partial t}|\,dx| \\ &\leq \|\nabla\rho\|\,\|\nabla\frac{\partial w}{\partial t}\| \\ &\leq c\|\nabla\rho\|^2+\frac{1}{6}\|\nabla\frac{\partial w}{\partial t}\|^2 \end{aligned} \tag{3.79}$$

Collecting estimates (3.73)-(3.79), we finally get an inequality of the form

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\Delta w\|^2+\|\Delta\rho\|^2+\|\nabla\Delta\rho\|^2)+\|\nabla\Delta\rho\|^2+\frac{1}{2}\|\nabla\frac{\partial w}{\partial t}\|^2 \\ &\leq c(\|u_1\|^\beta_{H^1(\Omega)}+\|u_2\|^\beta_{H^1(\Omega)}+1)\|u\|^2_{H^1(\Omega)}+c'\|\nabla\rho\|^2,\ \beta\geq 0. \end{aligned} \tag{3.80}$$

The addition of (3.70) and ϵ_3(3.80), for $\epsilon_3>0$ small enough, gives

$$\frac{dE_3}{dt}+cE_3+\|\frac{\partial w}{\partial t}\|^2+\|\nabla\frac{\partial w}{\partial t}\|^2\leq c'(\|u_1\|^\gamma_{H^2(\Omega)}+\|u_2\|^\gamma_{H^2(\Omega)}+1)\|u\|^2_{H^1(\Omega)},\ \gamma\geq 0. \tag{3.81}$$

where $E_3=E_2+\epsilon_3(\|\Delta w\|^2+\|\Delta\rho\|^2+\|\nabla\Delta\rho\|^2)$ satisfies

$$c(\|w\|^2_{H^2(\Omega)}+\|\rho\|^2_{H^3(\Omega)})\leq E_3\leq c'(\|w\|^2_{H^2(\Omega)}+\|\rho\|^2_{H^3(\Omega)}). \tag{3.82}$$

We deduce, thanks to Gronwall's lemma and (3.82), that

$$\|w(t)\|^2_{H^2(\Omega)}+\|\rho(t)\|^2_{H^3(\Omega)}\leq h(t)(\|u_0\|^2_{H^1(\Omega)}+\|\varphi_0\|^2_{H^2(\Omega)}),\ \forall t\in[0,T], \tag{3.83}$$

where $h(t)=c'\int_0^t e^{c(t-s)}(\|u_1\|^\gamma_{H^2(\Omega)}+\|u_2\|^\gamma_{H^2(\Omega)}+1)\,ds.$ □

Theorem 3.3. *The semi-group $S(t)$, $t\geq 0$, generated by the problem (3.9)-(3.12) is a Lipschitz in space and Holder in time on $[0,T]\times B$ where $B\subset H^2(\Omega)\times H^3(\Omega)$ is a bounded.*

Proof. The Lipschitz continuity in space being a consequence of the result on the uniqueness of solutions obtained in [], it just remains to prove a holder condition in time for the semigroup $S(t)$, $t\geq 0$. Let initial data belong to B, i.e. let $R>0$ be fixed, such that $\|u_0\|^2_{H^1(\Omega)}+\|\varphi_0\|^2_{H^2(\Omega)}\leq R$. Thus, for all $t_1\geq 0$ and $t_2\geq 0$, two different times, thanks to estimates on $\|\frac{\partial u}{\partial t}\|^2_{H^1(\Omega)}$ and $\|\frac{\partial\varphi}{\partial t}\|^2_{H^2(\Omega)}$ and Cauchy-Schwartz's inequality, we have

$$\begin{aligned} \|S(t_1)(u_0,\varphi_0)-S(t_2)(u_0,\varphi_0)\|_{H^1(\Omega)\times H^2(\Omega)} &= \|u(t_1)-u(t_2),\varphi(t_1)-\varphi(t_2)\|_{H^1(\Omega)\times H^2(\Omega)} \\ &= \|u(t_1)-u(t_2)\|_{H^1(\Omega)}+\|\varphi(t_1)-\varphi(t_2)\|_{H^2(\Omega)} \\ &\leq \int_{t_1}^{t_2}\|\frac{\partial u(\tau)}{\partial t}\|_{H^1(\Omega)}\,d\tau+\int_{t_1}^{t_2}\|\frac{\partial\varphi(\tau)}{\partial t}\|_{H^2(\Omega)}\,d\tau \\ &\leq c|t_1-t_2|^{\frac{1}{2}}, \end{aligned} \tag{3.84}$$

where c depends on $T>0$ and R.

□

We deduce from theorem 3.2 and theorem 3.3, the following result

Theorem 3.4. *The dynamical system* $(S(t), X)\, t \geq 0$*, associated to the problem (3.9)-(3.12) possesses an exponential attractor noted* $\mathcal{M}$ *in* X*.*

Remark 3.5. Compared with the global attractor, an exponential attractor is expected to be more robust under perturbations. Indeed, the rate of attraction of trajectories towards the global attractor can be slow and it is extremely difficult, even impossible, to estimate this rate of attraction according to the physical parameters of the problem in general. As a result of global attractor may change drastically under small perturbations.

Corollary 3.6. *The semigroup* $S(t)\, t \geq 0$*, possesses the finite-dimension global attractor* $\mathcal{A}$ *in* X*.*

Remark 3.7. The finite-dimensionality means, basically, that, although the initial phase space is infinite-dimensional, the reduced dynamics is, in an appropriate sense, finite-dimensional and can be described by a finite number of parameter.

4 Conclusion

In this work, we demonstrated that the attractor of the dynamical system we studied is of finite dimension by showing that the dynamical system has an exponential attractor. For the future, it would be interesting to determine precisely the value of the dimension of the attractor. There are many techniques for determining the dimension of an attractor. These include methods derived from the statistical theory of extreme values, whose estimates of the local fractal dimension are becoming increasingly reliable.

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of this manuscript.

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Competing Interests

Authors have declared that no competing interests exist.

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